The Basic Problem

Given a set of tabular values (available from an experiment) of a function f.

X	x_0	x_1	x_2	...	x_n
$f(X)$	$f(x_0)$	$f(x_1)$	$f(x_3)$	...	$f(x_n)$

Where the explicit nature of the function f is not known.

To find a function ϕ such that $\phi(x_i) = f(x_i)$ for all $0 \leq i \leq n$.

Such a function ϕ is called an *interpolating function.*

Geometrically

The problem is :

To find a function / polynomial whose graph passes through the given set of $(n+1)-$ points,

$$(x_0, f(x_0)), \quad (x_1, f(x_1)), \cdots, (x_n, f(x_n)).$$

Polynomial interpolation

If ϕ is a polynomial then the process is called polynomial interpolation and ϕ is called an interpolating polynomial.

Some types of interpolation

- Polynomial interpolation
- Piecewise Polynomial(Spline) interpolation
- Rational interpolation
- Trigonometric interpolation
- Exponential interpolation

We study : Polynomial and Piecewise polynomial interpolations.

Example

Find a polynomial which fits the following data.

X	x_0	x_1	x_2
$f(X)$	$f(x_0)$	$f(x_1)$	$f(x_2)$

Example

Solution:

$$P_3(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)}f(x_0) + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)}f(x_1)$$
$$+\frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}f(x_2).$$

This is a polynomial of degree 2.

- Called **Lagrange's** interpolation formula.

Lagrange's Interpolation formula

Similarly,

$$p_n(x) = \frac{(x-x_1)(x-x_2)\cdots(x-x_n)}{(x_0-x_1)(x_0-x_2)\cdots(x_0-x_n)}f(x_0)$$

$$+\frac{(x-x_2)(x-x_3)\cdots(x-x_n)}{(x_1-x_0)(x_1-x_2)\cdots(x_1-x_n)}f(x_1)$$

$$+\cdots+\frac{(x-x_0)(x-x_1)\cdots(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)\cdots(x_1-x_{n-1})}f(x_n)$$

is the " unique polynomial of degree $\leq n$" which interpolates f at the $n+1$ points $x_0, x_1, \cdots, x_n$.

Lagrange's Interpolation formula

If we write

$$l_i(x) = \frac{(x - x_0)(x - x_1)\cdots(x - x_{i-1})(x - x_{i+1})\cdots(x - x_n)}{(x_i - x_0)(x_i - x_1)\cdots(x_i - x_{i-1})(x_i - x_{i+1})\cdots(x_i - x_n)}$$

then

$$\boxed{p_n(x) = \sum_{i=0}^{n} l_i(x) f(x_i)}$$

is the " unique(how?) polynomial of degree $\leq n$" which interpolates f at the $n + 1$ points $x_0, x_1, \cdots, x_n$.

We proved:

Theorem 1. ***(Existence and Uniqueness theorem for interpolating polynomials.)*** *If $x_0, x_1, x_2, \cdots, x_n$ are $n+1$ distinct numbers and f is a function whose values are given at these numbers, then a unique polynomial $p(x)$ of* ***degree*** *$\leq n$ with $p(x_i) = f(x_i)$ for all $i = 1, 2, \cdots, n$.*

This polynomial is given by

$$\boxed{p(x) = \sum_{i=0}^{n} l_i(x) f(x_i)}$$

Thus we have:

Given $x_0, x_1, x_2, x_3, \cdots, x_n$ and $f(x_0), f(x_1), \cdots, f(x_n)$, we have:

Lagrange's polynomial	Degree at most	interpolates f at
$p_1(x)$	1	x_0, x_1
$p_2(x)$	2	x_0, x_1, x_2
$p_3(x)$	3	x_0, x_1, x_2, x_3
$\vdots$	$\vdots$	$\vdots$
$p_n(x)$	n	$x_0, x_1, x_2, x_3, \cdots, x_n$

Example

Find the interpolating polynomial in Lagrangian form for the data.

x	-2	-1	1	3
$f(x)$	-15	-4	0	20

Solution:

We calculate $p_3(x)$.

$$p_3(x) = l_0(x).f(x_0) + l_1(x).f(x_1) + l_2(x).f(x_2) + l_3(x).f(x_3)$$

Where $l_0(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} = \frac{(x+1)(x-1)(x-3)}{(-1)(-3)(-5)}$

$$= \frac{-1}{15}(x^3 - 3x^2 - x + 3)$$

Similarly,

$$l_1(x) = \frac{1}{8}(x^3 - 2x^2 - 5x + 6)$$
$$l_2(x) = \frac{-1}{12}(x^3 - 7x - 6)$$
$$l_3(x) = \frac{1}{40}(x^3 + 2x^2 - x + -2)$$

Therefore,

$$p_3(x) = -15\{\frac{-1}{15}(x^3 - 3x^2 - x + 3)\} - 4\{\frac{1}{8}(x^3 - 2x^2 - 5x + 6)\}$$

$$+0 + 20\{\frac{1}{40}(x^3 + 2x^2 - x + -2)\}$$
$$= x^3 - x^2 + x - 1.$$

Error: Let $f : [a, b] \longrightarrow \mathbb{R}$

$$E_n(x) = \frac{f^{n+1}}{n+1}(\xi)(x - x_0)(x - x_1)\cdots(x - x_n)$$

For some $\xi \in (a, b)$.

Problem: We do not know ξ, hence it would be difficult to compute error with this formula. However, this can be used in order to find the upper bound of the error, i,e., we have a number in hand such that error should not exceed by this number.

A disadvantage

• If more data become available, the work performed to generate the original Lagrange form cannot be reused to compute a higher-degree polynomial.

This can be rectified by writing the Lagrange's interpolation polynomial in **Newton's form**.

For this we use the idea of **divided differences**.

First order divided differences

$$f[x_0, x_1] = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

$$f[x_1, x_2] = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

$$\vdots \qquad \vdots$$

In general,

$$f[x_i, x_{i+1}] = \frac{f(x_{i+1}) - f(x_i)}{x_{i+1} - x_i}$$

$$\vdots \qquad \vdots$$

$$f[x_{n-1}, x_n] = \frac{f(x_n) - f(x_{n-1})}{x_n - x_{n-1}}$$

Second order divided differences

$$f[x_0, x_1, x_2] = \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0}$$

$$f[x_1, x_2, x_3] = \frac{f[x_2, x_3] - f[x_1, x_2]}{x_3 - x_1}$$

$$\vdots \qquad\qquad \vdots$$

$$f[x_i, x_{i+1}, x_{i+2}] = \frac{f[x_{i+1}, x_{i+2}] - f[x_i, x_{i+1}]}{x_{i+2} - x_i}$$

$$\vdots \qquad\qquad \vdots$$

$$f[x_{n-2}, x_{n-1}, x_n] = \frac{f[x_{n-1}, x_n] - f[x_{n-1}, x_n]}{x_n - x_{n-2}}$$

Divided differences of n^{th}(≥ 3) order

- Can be defined recursively in a similar way.

$$f[x_i, x_{i+1}, \cdots, x_{i+n}] = \frac{f[x_{i+1,\cdots,x_{i+n}}] - f[x_i, x_{i+1,\cdots,x_{i+n-1}}]}{x_{i+n} - x_i}$$

The numbers $f(x_0), f(x_1), \cdots, f(x_n)$ are called divided differences of 0**-th order** and are denoted by $f[x_0], f[x_1], \cdots, f[x_n]$ respectively.

We say: $f[x_0, x_1, \cdots, x_n]$ as the divided difference of f at the points $x_0, x_1, \cdots, x_n$.

Divided differences in Tabular form (when n= 3.)

x	$f(x)$	First order	Second order	Third order
x_0	$f[x_0]$			
		$f[x_0, x_1]$		
x_1	$f[x_1]$		$f[x_0, x_1, x_2]$	
		$f[x_1, x_2]$		$f[x_0, x_1, x_2, x_3]$
x_2	$f[x_2]$		$f[x_1, x_2, x_3]$	
		$f[x_2, x_3]$		
x_3	$f[x_3]$			

Example

Consider the data:

x	-2	-1	1	3
$f(x)$	-15	-4	0	20

The divided difference table:

x	$f(x)$	$f[,]$	$f[,,]$	$f[,,,]$
-2	-15			
		11		
-1	-4		-3	
		2		1
1	0		2	
		10		
3	20			

The Newton's form of a polynomial

$$p_n(x) = A_0 + A_1(x-x_0) + A_2(x-x_0)(x-x_1) + \cdots + A_n(x-x_0)(x-x_2)\cdots(x-x_{n-1})$$

We need to compute $A_i, i = 0, 1 \cdots n$.

This can be done easily by using

$$f(x_i) = p_n(x_i), i = 0, 1 \cdots n$$

Can you see a relation between A_i and divided difference?

Idea

We prove:

$$A_k = f[x_0, x_1, \cdots, x_k]$$

for all $k = 0, 1, 2, \cdots, n.$

Thus the Newton's form of $p_n(x)$ is:

$$p_n(x) = f[x_0]$$

$$+f[x_0, x_1](x - x_0)$$

$$+f[x_0, x_1, x_2](x - x_0)(x - x_1)$$

$$+f[x_0, x_1, x_2, x_3](x - x_0)(x - x_1)(x - x_2)$$

$$\cdots + f[x_0, x_1, \cdots, x_n](x - x_0)(x - x_1)(x - x_2)\cdots(x - x_{n-1})$$

- *Look at the divided difference table and recognize the coefficients.*

- Thus the divided difference table helps us to write the interpolation polynomial easily.

x	$f(x)$	First order	Second order	Third order
x_0	$f[x_0]$			
		$f[x_0, x_1]$		
x_1	$f[x_1]$		$f[x_0, x_1, x_2]$	
		$f[x_1, x_2]$		$f[x_0, x_1, x_2, x_3]$
x_2	$f[x_2]$		$f[x_1, x_2, x_3]$	
		$f[x_2, x_3]$		
x_3	$f[x_3]$			

Example

Consider the data:

x	-2	-1	1	3
$f(x)$	-15	-4	0	20

The divided difference table:

x	$f(x)$	$f[,]$	$f[,,]$	$f[,,,]$
-2	-15			
		11		
-1	-4		-3	
		2		1
1	0		2	
		10		
3	20			

$$p_3(x) = -15 + 11(x+2) - 3(x+2)(x+1) + 1(x+2)(x+1)(x-1)$$

$$= x^3 - x^2 + x - 1.$$

When the points $x_0, x_1, \cdots, x_n$ are equispaced.

• While forming the divided difference table, we look for the correct division, and there is chance of mistake. Did you realize this?. Let $x_{i+1} - x_i = h$ **for all** $i = 0, 1, 2, \cdots, n-1.$

• The Newton's divided difference formula can be further simplified to Newton's forward difference form.

We first define the forward differences.

Forward differences

First order forward differences:

$$\triangle^1 f(x_0) = f(x_1) - f(x_0)$$

$$\triangle^1 f(x_1) = f(x_2) - f(x_1)$$

$$\vdots \qquad \vdots$$

$$\triangle^1 f(x_i) = f(x_{i+1}) - f(x_i)$$

$$\vdots \qquad \vdots$$

$$\triangle^1 f(x_{n-1}) = f(x_n) - f(x_{n-1})$$

We call $\triangle^1 f(x_i)$ as: The first order forward difference of f at x_i.

Forward differences of higher order:

- can be computed using the recursive following formula.

If $n > 1$ then we define,

$$\triangle^n f(x_i) = \triangle^{n-1} f(x_{i+1}) - \triangle^{n-1} f(x_i)$$

Forward difference table

Consider the data:

x	-2	-1	0	1
$f(x)$	-15	-4	0	20

The forward difference table:

x	$f(x)$	$\triangle^1 f$	$\triangle^2 f$	$\triangle^3 f$
-2	-15			
		11		
-1	-4		-7	
		4		23
0	0		16	
		20		
1	20			

Divided differences in terms of forward differences

Suppose $f : [a, b] \to \mathbb{R}$ is given at the $n + 1$ equispaced points $x_0, x_1, \cdots, x_n$ with spacing $h = x_{i+1} - x_i$.

$$f[x_0, x_1] = \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \frac{1}{h}\triangle f(x_0)$$

$$\begin{aligned} f[x_0, x_1, x_2] &= \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0} \\ &= \frac{1}{2h}\Big[\frac{\triangle f(x_1) - \triangle f(x_0)}{h}\Big] = \frac{1}{2h^2}\triangle^2 f(x_0) \end{aligned}$$

In general,

$$f[x_0, x_1, \cdots, x_k] = \frac{1}{k!\ h^k}\triangle^k f(x_0).$$

Newton's forward difference formula.

Suppose $f : [a, b] \to \mathbb{R}$ is given at the $n + 1$ equispaced points $x_0, x_1, \cdots, x_n$ with spacing $h = x_{i+1} - x_i$.

Given x, let $s = \frac{x - x_0}{h}$.

Thus x can be written in the form $x = x_0 + sh$ for some s

In particular, $x_i = x_0 + ih$ for all $i = 0, 1, 2, \cdots, n$.

Thus

$$x - x_i = x_0 + sh - (x_0 + ih) = (s - i)h$$

Now the interpolating polynomial $p_n(x)$ can be written as:

$$p_n(x) = p_n(x_0 + sh) = f[x_0]$$

$$+sh\ f[x_0, x_1]$$

$$+s(s-1)h^2\ f[x_0, x_1, x_2]$$

$$+s(s-1)(s-2)h^3 f[x_0, x_1, x_2, x_3]$$

$$\cdots +\ s(s-1)(s-2)\cdots(s-n+1)f[x_0, x_1, \cdots, x_n]$$

$$= \sum_{k=0}^{n}\ s(s-1)(s-2)\cdots(s-k+1)f[x_0, x_1, \cdots, x_k]$$

$$= \sum_{k=0}^{n} \binom{s}{k} \; k! \, h^k \; f[x_0, x_1, \cdots, x_k]$$

$$= \sum_{k=0}^{n} \binom{s}{k} \triangle^k f(x_0)$$

Where

$$\binom{s}{k} = \frac{s(s-1)\cdots(s-k+1)}{k!}.$$

Example

Prepare the forward difference table for the following data. Using Newton's forward interpolating polynomial , find approximate value of $f(0.1)$.

x	0	0.2	0.4	0.6	0.8
$f(x)$	0.12	0.46	0.74	0.90	1.2

x	$f(x)$	$\triangle^1 f$	$\triangle^2 f$	$\triangle^3 f$	$\triangle^4 f$
0	0.12				
		0.34			
0.2	0.46		-0.06		
		0.28		-0.06	
0.4	0.74		-0.12		0.32
		0.16		0.26	
0.6	0.90		0.14		
		0.30			
0.8	1.2				

To interpolate at $x = 0.1$

$$s = \frac{x - x_0}{h} = \frac{0.1 - 0}{0.2} = \frac{1}{2}.$$

Therefore the binomial coefficients are:

$$\binom{s}{0} = 1, \quad \binom{s}{1} = \frac{1}{2}, \qquad \binom{s}{2} = \frac{\frac{1}{2}(\frac{1}{2} - 1)}{1.2} = -\frac{1}{8}.$$

$$\binom{s}{3} = \frac{s(s-1)(s-3)}{1.2.3} = \frac{\frac{1}{2}(-\frac{1}{2})(-\frac{3}{2})}{1.2.3} = \frac{1}{16}.$$

$$\binom{s}{4} = \frac{s(s-1)(s-2)(s-3)}{1.2.3.4} = \frac{\frac{1}{2}(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})}{1.2.3.4} = -\frac{5}{128}.$$

Therefore

$$p_4(x_0+sh) = f(x_0)+s\triangle^1 f(x_0)+\binom{s}{2}\triangle^2 f(x_0)+\binom{s}{3}\triangle^3 f(x_0)+\binom{s}{4}\triangle^4 f(x_0).$$

$$= 0.12 + \frac{1}{2}(0.34) + \frac{-1}{8}(-0.06) + \frac{1}{16}(-0.06) + \frac{-5}{128}(0.32) = 0.28125$$

Hence $f(0.1) \approx 0.28125$

Piece-wise polynomial interpolation

• Polynomials with higher degree grow / decay very fast.

• When the number of tabular points is more in number, the interpolating polynomial would be of higher degree and hence there is a chance of getting more error.

Therefore,

We divide the given interval into small subintervals and on each small subinterval we fit a polynomial of lower degree.

Newton's backward difference formula

$$\nabla f(x_1) = f(x_1) - f(x_0)$$

$$\nabla f(x_2) = f(x_2) - f(x_1)$$

$$\nabla f(x_i) = f(x_i) - f(x_{i-1})$$

$$\nabla f(x_n) = f(x_n) - f(x_{n-1})$$

meaning full for all $i = 1, 2, \cdots, n$

Note that

$$\triangle^1 f(x_2) = f(x_3) - f(x_2) = \nabla^1 f(x_3)$$

Thus in general,

$$\triangle^1 f(x_i) = \nabla^1 f(x_{i+1})$$

for all $i = 1, 2, \cdots, n-1$

Thus we can use the forward difference table to determine the backward differences.

Newton's backward difference formula

Given f at the equispaced points $x_0, x_1, \cdots, x_n$ with spacing h.

Let x be in between x_0 and x_n.

In this case, we take $s = \frac{x-x_n}{h}$

Now

$$p_n(x) = f[x_n]$$

$$+f[x_n, x_{n-1}](x - x_n)$$

$$+f[x_n, x_{n-1}, x_{n-2}](x - x_n)(x - x_{n-1})$$

$$+f[x_n, x_{n-1}, x_{n-2}, x_{n-3}](x - x_n)(x - x_{n-1})(x - x_{n-2})$$

$$\cdots + f[x_n, x_{n-1}, \cdots, x_0](x - x_n)(x - x_{n-1})(x - x_{n-2})\cdots(x - x_1)$$

$$= f[x_n] \quad + sh\ f[x_n, x_{n-1}] \quad + s.(s+1).h^2\ f[x_n, x_{n-1}, x_{n-2}]$$

$$+ s.(s+1).(s+2).h^3 f[x_n, x_{n-1}, x_{n-2}, x_{n-3}]$$

$$\cdots + s.(s+1).\cdots(s+n-1).h^n.f[x_n, x_{n-1}, \cdots, x_0]$$

We can now prove that:

$$f[x_n, x_{n-1}, \cdots, x_{n-k}] = \frac{1}{k!\ h^k} \nabla^k f(x_n).$$

Thus

$$p_n(x) = f[x_n] + s.\nabla^1 f(x_n) + \frac{s(s+1)}{2}\nabla^2 f(x_n) +$$

$$\cdots + \frac{s(s+1)(s+2)\cdots(s+n-1)}{n!}\nabla^n f(x_n)$$

If we write

$$\binom{-s}{k} = \frac{-s(-s-1)(-s-2)\cdots(-s-k+1)}{k!}$$

$$= (-1)^k \frac{s(s+1)(s+2)\cdots(s+k-1)}{k!}$$

then

$$p_n(x) = f[x_n] + \sum_{k=1}^{n} (-1)^k \binom{-s}{k} . \nabla^k f(x_n)$$

Example

Interpolate at $x = 0.65$ using Newton's backward interpolation formula.

x	0	0.2	0.4	0.6	0.8
$f(x)$	0.12	0.46	0.74	0.90	1.2

x	$f(x)$	$\triangle^1 f$	$\triangle^2 f$	$\triangle^3 f$	$\triangle^4 f$
0	0.12				
		0.34			
0.2	0.46		-0.06		
		0.28		-0.06	
0.4	0.74		-0.12		**0.32**
		0.16		**0.26**	
0.6	0.90		**0.14**		
		0.30			
0.8	**1.2**				

Give point $x = 0.65$.

In the case of Newton's backward form:

$$s = \frac{x - x_n}{h} = \frac{x - x_4}{h} = \frac{0.65 - 0.8}{0.2} = -\frac{3}{4}.$$

The binomial coefficients :

$$\binom{-s}{0} = 1, \quad \binom{-s}{1} = \frac{3}{4}, \quad \binom{-s}{2} = \frac{-3}{32}$$

$$\binom{-s}{3} = \frac{5}{128}, \quad \binom{-s}{4} = \frac{-45}{2048}$$

$$p_4(x) = f[x_4] + \sum_{k=1}^{4} (-1)^k \binom{-s}{k} . \nabla^k f(x_4)$$

$$f(0.65) = p_4(0.8 - \frac{3}{4} \times 0.2)$$

$$= f[x_0] - \frac{3}{4}\nabla^1(x_4) - \frac{3}{32}\nabla^2(x_4) - \frac{5}{128}\nabla^3(x_4) - \frac{45}{2048}\nabla^4(x_4)$$

$$= 0.944687$$

We Use the interpolating polynomial

- **To estimate the missing data.**

OR

- **To approximate f at some intermediate points.**