ELEMENTARY TRANSFORMATIONS (OR OPERATIONS)

Any one of the following operations on a matrix is called an elementary transformation. (or E-operation).

(i) Interchange of two rows or two columns.

The interchange of i^{th} and j^{th} rows is denoted by R_{ij}.

The interchange of i^{th} and columns is denoted by C_{ij}.

(ii) Multiplication of (each element of) a row or column by a non-zero number k.

The multiplication of i^{th} row by k is denoted by $R_i(k)$.

The multiplication of i^{th} column by k is denoted by $C_i(k)$.

(iii) Addition of k times the elements of a row (or column) to the corresponding elements of another row (or column), $k \neq 0$,

The addition of k times the j^{th} row to the i^{th} row is denoted by $R_{ij}(k)$.

The addition of k times the i^{th} column is denoted by $C_{ij}(k)$.

If a matrix B is obtained from a matrix A by one or more E-operations, then B is said to be equivalent to A. Two equivalent matrices A and B are written as A - B.

ELEMENTARY MATRICES

The matrix obtained from a unit matrix I by subjecting it to one the E-operations is called an elementary matrix.

$e.g.$ Let $I \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Operating R_{23} or C_{23} on I, we get the elementary matrix $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$. It is denoted by E_{23}

THEOREMS ON THE EFFECT OF E-OPERATIONS ON MATRICES

(a) Any E-row operation on the product of two matrices is equivalent to the same E-row operation on the pre-factor.

If the E-row operation is denoted by R, then R(AB) = R(A).B.

(b) Any E-column operation on the product of two matrices is equivalent to the same E-column operation on the post-factor.

(c) Every E-row operation on a matrix is equivalent to pre-multiplication by the corresponding E-matrix.

Thus, the effect of E-row operation R_{ij} on $A = E_{ij} . A$

The effect of E-row operation $R_i(k)$ on $A = E_i(k).\ A$

The effect of E-row operation $R_{ij}(k)$ on $A = A. E_{ij}(k)$.

(e) The inverse of an elementary matrix is an elementary matrix.

EXERCISE

Example 1. Transform $\begin{bmatrix} 1 & 3 & 3 \\ 2 & 4 & 10 \\ 3 & 8 & 4 \end{bmatrix}$ into a unit matrix by using elementary transformations.

Sol. We have $$\begin{bmatrix} 1 & 3 & 3 \\ 2 & 4 & 10 \\ 3 & 8 & 4 \end{bmatrix}$$

Operating $R_{21}(-2), R_{31}(-3)$

$$-\begin{bmatrix} 1 & 3 & 3 \\ 0 & -2 & 4 \\ 0 & -1 & -5 \end{bmatrix}$$

Operating $R_2\left(-\frac{1}{2}\right)$

$$-\begin{bmatrix} 1 & 3 & 3 \\ 0 & 1 & -2 \\ 0 & -1 & -5 \end{bmatrix}$$

Operating $R_{12}(-3), R_{32}(1)$

$$-\begin{bmatrix} 1 & 0 & 9 \\ 0 & 1 & -2 \\ 0 & 0 & -7 \end{bmatrix}$$

Operating $R_3\left(-\frac{1}{7}\right)$

$$-\begin{bmatrix} 1 & 0 & 9 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$

Operating $R_{13}(-9)R_{23}(2)$

$$-\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}$$

INVERSE OF MATRIX BY E-OPERATIONS (Gauss-Jordan Method)

The elementary row transformations which reduce a square matrix A to the unit matrix, when applied to the unit matrix, gives the inverse matrix A^{-1}.

To compute the inverse of a matrix, we use the concept of equivalent matrices.

If we are to find out the inverse of a non-singular square matrix A, we first write A as equivalent to I, a unit matrix of the same order.

$$A - I$$

Then we apply elementary row operations on them. The objective is to reduce A to I, As soon as this is achieved, the other matrix gives A^{-1}.

$$I - A^{-1}$$

This is an elegant way of determining the inverse or reciprocal of a matrix A.

Example 3. *Employing elementary row transformations, find the inverse of the matrix.*

$$A = \begin{bmatrix}0 & 1 & 2\\1 & 2 & 3\\3 & 1 & 1\end{bmatrix}$$

Sol. Let $A = I$

$$\begin{bmatrix}0 & 1 & 2\\1 & 2 & 3\\3 & 1 & 1\end{bmatrix} - \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}$$

Operating R_{12}

$$\begin{bmatrix}1 & 2 & 3\\0 & 1 & 2\\3 & 1 & 1\end{bmatrix} - \begin{bmatrix}0 & 1 & 0\\1 & 0 & 0\\0 & 0 & 1\end{bmatrix}$$

Operating $R_{31}(-3)$

$$\begin{bmatrix}1 & 2 & 3\\0 & 1 & 2\\0 & -5 & -8\end{bmatrix} - \begin{bmatrix}0 & 1 & 0\\1 & 0 & 0\\0 & -3 & 1\end{bmatrix}$$

Operating $R_{12}(-2), R_{32}(5)$

$$\begin{bmatrix}1 & 0 & -1\\0 & 1 & 2\\0 & 0 & 2\end{bmatrix} - \begin{bmatrix}-2 & 1 & 0\\1 & 0 & 0\\5 & -3 & 1\end{bmatrix}$$

Operating $R_{12}\left(\frac{1}{2}\right), R_{23}(-1)$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} - \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -4 & 3 & -1 \\ 5 & -3 & 1 \end{bmatrix}$$

Operating $R_3\left(\frac{1}{2}\right)$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -4 & 3 & -1 \\ 5/2 & -3/2 & 1/2 \end{bmatrix}$$

$I - A^{-1}$

$$A^{-1} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -4 & 3 & -1 \\ 5/2 & -3/2 & 1/2 \end{bmatrix}$$

Exercise

1. Reduce the matrix A to triangular form, $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 7 \\ 3 & 1 & 2 \end{bmatrix}$.

2. If X, Y are non-singular matrices and $B = \begin{bmatrix} X & O \\ O & Y \end{bmatrix}$, show that $B^{-1} = \begin{bmatrix} X^{-1} & O \\ O & Y^{-1} \end{bmatrix}$ where O is a null matrix.

3. Find the inverse of the matrix $\begin{bmatrix} 1 & 0 & k \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & s \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & p & 0 \\ 0 & 0 & 1 \end{bmatrix}$ without first evaluating the product.

4. Find the inverse of the following matrices by using elementary row operations:

(i) $\begin{bmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix}$ (ii) $\begin{bmatrix} 2 & 1 & -1 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ (iii) $\begin{bmatrix} 8 & 4 & 3 \\ 2 & 1 & 1 \\ 1 & 2 & 1 \end{bmatrix}$

(iv) $\begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix}$ (v) $\begin{bmatrix} 1 & 2 & 1 \\ 3 & 2 & 3 \\ 1 & 1 & 2 \end{bmatrix}$ (vi) $\begin{bmatrix} 1 & 0 & -1 \\ 3 & 4 & 5 \\ 0 & -6 & -7 \end{bmatrix}$

5. Employing elementary row transformations, find the inverse of the following non-singular matrix.

(i) $\begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & 3 & 3 & 2 \\ 2 & 4 & 3 & 3 \\ 1 & 1 & 1 & 1 \end{bmatrix}$ (ii) $\begin{bmatrix} 2 & 1 & -1 & 2 \\ 1 & 3 & 2 & -3 \\ -1 & 2 & 1 & -1 \\ 2 & -3 & -1 & 4 \end{bmatrix}$

Answer

1. $\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & -2 \end{bmatrix}$ 2. $\begin{bmatrix} 1 & 0 & -k \\ 0 & 1/p & -8/p \\ 0 & 0 & 1 \end{bmatrix}$

4. (i) $\begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix}$ (ii) $\begin{bmatrix} 8 & -1 & -3 \\ -5 & 1 & 2 \\ 10 & -1 & -4 \end{bmatrix}$ (iii) $\begin{bmatrix} 1 & -2 & -1 \\ 1 & -5 & 2 \\ -3 & 12 & 0 \end{bmatrix}$

(iv) $\begin{bmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix}$ (v) $\begin{bmatrix} -1/4 & 3/4 & -1 \\ 3/4 & -1/4 & 0 \\ -1/4 & -1/4 & 1 \end{bmatrix}$ (vi) $\begin{bmatrix} 1/10 & 3/10 & 1/5 \\ 21/20 & -7/20 & -2/5 \\ -9/10 & -3/10 & 1/5 \end{bmatrix}$

5. (i) $\begin{bmatrix} 1 & -2 & 1 & 0 \\ 1 & -2 & 2 & -3. \\ 0 & 1 & -1 & 1 \\ -2 & 3 & -2 & 3 \end{bmatrix}$ (i) $\begin{bmatrix} 2 & 5 & -7 & 1 \\ 5 & -1 & 5 & -2 \\ -7 & 5 & 11 & 10 \\ 1 & -2 & 10 & 5 \end{bmatrix}$