MATRICES

A matrix is defined as a rectangular array (of arrangement in rows or columns) of scalars subject to certain rules of operations.

If mn numbers (real or complex) or functions are arranged in the form of a rectangular array A having m rows (horizontal lines) and n columns (vertical lines) then A is called an $m \times n$ matrix. Each of the mn numbers is called an element of the matrix.

There are different notations of enclosing the elements constituting a matrix $viz.$ [ ], ( ). ‖ ‖ but square bracket is generally used, An $m \times n$ matrix is also called a matrix of order $m \times n$.

For example, $\begin{bmatrix} 2 & -1 & 5 \\ 3 & 0 & 4 \end{bmatrix}$ is a 2×3 matrix or matrix of order 2×3. It has two rows and three columns. The numbers $2, -1, 5, 3, 0, 4$ are its elements.

An $m \times n$ matrix is usually written as $\begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots a_{2n} \\ a_{31} & a_{32} & a_{33} & \dots a_{3n} \\ \dots & \dots & \dots & \dots \dots . \\ a_{m1} & a_{m2} & a_{m3} & a_{mn} \end{bmatrix}$

Her each element has two suffixes. The first suffix indicates the row and the second suffix indicates the column in which the element lies. Thus, a_{23} is the element lying in the second row and third column, a_{ij} is the element lying in the i^{th} row and j^{th} column.

For brevity, a matrix is usually denoted by single capital letter A, B or C etc.

Thus, an $m \times n$ matrix A may be written as

$$A = [a_{ij}]_{m \times n} \text{ or } a = [a_{ij}], \text{ Where } i = 1, 2, 3, \dots \dots ., m; \ j = 1, 2, 3, \dots . n$$

APPLICATION OF MATRICES

In Algebra, the matrices have their largest application in the theory of simultaneous equations and linear transformations. $e.g.$, the set of simultaneous equations,

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$$

$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$$

may be symbolically represented by the equation,

AX=B

Where n $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \ B = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$

The theory of matrices has found of great utility in many branches of higher mathematics such as algebraic and differential equations, astronomy, mechanics, theory of electrical circuits, quantum mechanics, nuclear physics and aerodynamics.

Note. There is a similarity in the way of writing a determinant and a matrix but the two are entirely different. A determinant has a numerical value but a matrix has no numerical value. Matrix is only a convenient way of storing information.

TYPES OF MATRICES

(1) **Real Matrix**. A matrix is said to be real if all its elements are real numbers.

$e.g.,$ $\begin{bmatrix} \sqrt{5} & -3 & 1 \\ 0 & -\sqrt{2} & 7 \end{bmatrix}$ is a real matrix.

(2) **Square Matrix**. A matrix in which the number of rows is equal to the number of columns is called a **square matrix**, otherwise, it is said to be a **rectangular matrix**.

Thus, a matrix $A = [a_{ij}]_{m\times n}$ is a square matrix if $m = n$ and a rectangular matrix if $m \neq n$. "an n-rowed square matrix",

$e.g.,$ $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ is a 3 -rowed square matrix.

The elements a_{11}, a_{22}, a_{33} of a square matrix are called its **diagonal elements** and the diagonal along which these elements lie is called the **principal** or **leading diagonal**.

In a square matrix $A = [a_{ij}]$,

(i) for elements along the principal diagonal, $i = j$

(ii) for elements above the principal diagonal, $i < j$

(iii) for elements below the principal, $i > j$

(iv) for non-diagonal elements, $i \neq j$

The sum of the diagonal elements of a square matrix is called its trace or spur. Thus,

trace of the n rowed square matrix A= $[a_{ij}]$ is $a_{11} + a_{22} + a_{33} + \ldots + a_{nn} = \sum_{i=1}^{n} a_{ij}$

(3) **Row Matrix**. A matrix having only one row and any number of columns $i.e.,$ a matrix of order $1 \times n$ is called a **row matrix**. $e.g.,$ $[2 \;\; 5 - 3 \;\; 0]$ is row matrix.

(4) **Column Matrix**. A matrix having only one column and any number of rows $i.e.,$ a matrix of order $m \times 1$ is called a **column matrix**. $e.g.,$ $\begin{bmatrix} \sqrt{2} \\ 0 \\ -1 \end{bmatrix}$ is a column matrix.

(5) **Null Matrix**. A matrix in which each element is zero is called a **null matrix** or **void matrix** or a **zero matrix**, A null matrix of order $m \times n$ is denoted by $O_{m\times n}$.

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} = O_{3\times 2}, \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = O_{2\times 4}$$

(6) **Sub-matrix**. A matrix obtained from a given matrix A by deleting some of its rows or columns or both is called a **sub-matrix of A**.

Thus, $B = \begin{bmatrix} 3 & 0 \\ 1 & 4 \end{bmatrix}$ is a sub- matrix of $A = \begin{bmatrix} 0 & -1 & 2 & 5 \\ 3 & 5 & 0 & 7 \\ 1 & 6 & 4 & -2 \end{bmatrix}$

obtained by deleting the first row, second and fourth columns of A.

(7) **Diagonal Matrix**, A square matrix in which all non-diagonal elements are zero is called a **diagonal matrix**.

Thus, $A = [a_{ij}]_{n\times n}$ is a diagonal matrix if $a_{ij} = 0$ for $i \neq j$.

For example, $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ is a diagonal matrix.

An n −rowed diagonal matrix is briefly written as diag. $[d_1,\ d_2,\ \dots,\ d_n]$, where $d_1,\ \ d_2,\ \ d_n$ are the diagonal elements. Thus the above diagonal matrix A can be written as diag. $[2, -1,\ 0]$.

(8) **Scalar Matrix**. A diagonal matrix in which all the diagonal elements are equal to scalar, say k, is called a **scalar matrix**.

Thus a scalar matrix is a square matrix in which all non-diagonal elements are zero and all diagonal elements are equal to a scalar, say k.

$i.e.,$ $A = [a_{ij}]_{n\times n}$ is a scalar matrix if $a_{ij} = \begin{cases} 0 & when\ i \neq j \\ k & when\ i = j \end{cases}$

For example, $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} -5 & 0 & 0 & 0 \\ 0 & -5 & 0 & 0 \\ 0 & 0 & -5 & 0 \\ 0 & 0 & 0 & -5 \end{bmatrix}$ are scalar matrices.

(9) **Unit Matrix or Identity Matrix**. A scalar matrix in which each diagonal element is unity ($i.e.,$ 1) is called a **unit matrix or identity matrix**.

Thus, a unit matrix is a square matrix in which all non-diagonal elements are zero and all diagonal elements are equal to1.

$i.e.,$ $A = [a_{ij}]_{n\times n}$ is a unit matrix if $a_{ij} = \begin{cases} 0 & when\ \ i \neq j \\ 1 & when\ \ i = j \end{cases}$

A unit matrix of order n is denoted by I_n. If the order is evident, it may be simply denoted by I.

Thus, $\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} = I_2, \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix} = I_3$

(10) **Upper Triangular Matrix**. A square matrix in which all the elements below the principal diagonal are zero is called an **upper triangular matrix**.

Thus, $A = [a_{ij}]_{n\times n}$ is an upper triangular matrix if $a_{ij} = 0$ for $i > j$.

For example, $\begin{bmatrix}2 & 3 & 4\\0 & -1 & 5\\0 & 0 & 3\end{bmatrix}$ is an upper triangular matrix.

(11) **Lower Triangular Matrix**. A square matrix in which all the elements above the principal diagonal are zero is called a lower triangular matrix.

Thus $A - [a_{ij}]_{n\times n}$ is a lower triangular matrix if $a_{ij} = 0$ for $i < j$.

For example, $\begin{bmatrix}-1 & 0 & 0\\5 & 6 & 0\\3 & 2 & 0\end{bmatrix}$ is a lower triangular matrix.

(12) Triangular Matrix. A square matrix in which all the elements either below or above the principal diagonal are zero is called a triangular matrix. Thus, a triangular matrix is either upper triangular or lower triangular.

(13) **Equal Matrices**. Two matrices A and B are said to be equal (written as A =B) if and only if they have the same order and their corresponding elements are equal.

Thus, if $A = [a_{ij}]_{m\times n}$ and $B = [b_{ij}]_{p\times q}$, then A = B if and only if

(i) $m = p$ and $n = q$ (ii) $a_{ij} = b_{ij}$ for all i and j

(14) **Single Element Matrix**. A matrix having only one element is called a single element matrix. Thus any matrix [3] is a single matrix.

(15) **Singular and Non-singular Matrices**. A square matrix A is said to be singular if $|A| = 0$ and non-singular if $|A| \neq 0$.

For example $A = \begin{bmatrix}2 & -3 & 4\\2 & -3 & 4\\0 & -1 & 1\end{bmatrix}$ is a singular matrix since $|A| = 0$.

(16) **Tridiagonal Matrix**. Tridiagonal matrix is the matrix having non-zero entries only in the leading diagonal, sub-diagonal and super diagonal. In other words, we can define it as:

A real matrix $A = [a_{ij}]$ is said to be tridiagonal if $a_{ij} = 0$ for $|i - j| > 1$

For example, $A = \begin{bmatrix}a_{11} & a_{12} & 0 & 0\\a_{21} & a_{22} & a_{23} & 0\\0 & a_{32} & a_{33} & a_{34}\\0 & 0 & a_{43} & a_{44}\end{bmatrix}$ is a tridiagonal matrix.

(17) **Stochastic Matrix.** A square matrix A is called a stochastic matrix if all its elements are non-negative and the sum of the elements in each row is 1. Thus, the matrix

$A = [a_{ij}]_{n \times n}$ will be a stochastic matrix if $a_{ij} \geq 0$ for $0 \leq i \leq n, 0 \leq j \leq n$ and $\sum_{j=1}^{n} a_{ij} = 1$ for $i = 1, 2, \ldots, n.$ $e.g.,$ $A = \begin{bmatrix} 2/3 & 1/3 \\ 1/4 & 3/4 \end{bmatrix}$ is a stochastic matrix.

ADDITION AND SUBTRACTION OF MATRICES:

Two matrices are said to be conformable for addition and subtraction if they have the same order.

If A and B are two matrices of the same order, then their sum A + B is a matrix each element of which is obtained by adding the corresponding elements of A and B.

In general, if $A = [a_{ij}]_{m \times n}$ and $B = [b_{ij}]_{m \times n}$, then $A + B = C = [c_{ij}]_{m \times n}$, where $c_{ij} = a_{ij} + b_{ij}$, similarly, if A and B are two matrices of the same order, then their difference $A - B$ is a matrix whose elements are obtained by subtracting the elements of B from the corresponding elements of A.

In general, if $A = [a_{ij}]_{m \times n}$ and $B = [b_{ij}]_{m \times n}$, then $A - B = C = [c_{ij}]_{m \times n}$, where $c_{ij} = a_{ij} - b_{ij}$,

For example, if $A = \begin{bmatrix} 2 & 5 & -1 \\ 3 & 0 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -6 & 2 \\ -2 & 5 & 7 \end{bmatrix}$

then $$A + B = \begin{bmatrix} 2+1 & 5-6 & -1+2 \\ 3-2 & 0+5 & 4+7 \end{bmatrix} = \begin{bmatrix} 3 & -1 & 1 \\ 1 & 5 & 11 \end{bmatrix}$$

and $$A - B = \begin{bmatrix} 2-1 & 5-(-6) & -1+2 \\ 3-(-2) & 0-5 & 0-5 \end{bmatrix} = \begin{bmatrix} 1 & 11 & -3 \\ 5 & -5 & -3 \end{bmatrix}$$

MULTIPLICATION OF A MATRIX BY A SCALAR:

The product of a matrix $A = [a_{ij}]$ by a scalar k is denoted by kA and is obtained by multiplying every element of A by *k*.

Thus, $$kA = [ka_{ij}]$$

If $$A = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix}, \text{ then } kA = \begin{bmatrix} ka_1 & ka_2 & ka_3 \\ kb_1 & kb_2 & kb_3 \end{bmatrix}$$

In particular, $$-A = (= 1)A = \begin{bmatrix} -a_1 & -a_2 & -a_3 \\ -b_1 & -b_2 & -b_3 \end{bmatrix},$$

PROPERTIES OF MATRIX ADDITION:

(i) Matrix addition is commutative $i.e.,$ $A + B = B + A$

(ii) Matrix addition is associative $i.e.,$ $(A + B) + C = A + (B + C)$

(iii) For any matrix A, there exists a null matrix O of the same order as A such that

$$A + O = O + A = A.$$

(iv) For any matrix A, there exists a matrix -A of the same order as A such that

$$A + (-A) = (-A) + A = O$$

Illustrative Examples

Example 1. Find x, y, z and w if $3\begin{bmatrix} x & y \\ z & w \end{bmatrix} = \begin{bmatrix} x & 6 \\ -1 & 2w \end{bmatrix} + \begin{bmatrix} 4 & x+y \\ z+w & 3 \end{bmatrix}$

Sol. The given equation is $\begin{bmatrix} 3x & 3y \\ 3z & 3w \end{bmatrix} = \begin{bmatrix} x+4 & 6+x+y \\ -1+z+w & 2w+3 \end{bmatrix}$

Equation the corresponding elements on the two sides.

$$3x = x + 4, \quad 3y = 6 + x + y, \quad 3z = -1 + z + w, \quad 3w = 2w + 3$$

$$\Rightarrow \quad 2x = 4, \quad 2y = 6 + x, \quad 2z = -1 + w, \quad w = 3$$

$$\Rightarrow \quad x = 2, \quad y = 4, \quad z = 1, \quad w = 3,$$

MATRIX MULTIPLICATION:

Two matrices A and B are said to be conformable for the product AB (**in this very order of A and B**) if the number of columns in A (called the pre-factor) is equal to the number of rows in B (called the post-factor).

Thus, if the orders of A and B are $m \times n$ and $p \times q$ respectively, then

(i) AB is defined if number of columns in A = number of rows in B, $i.e.$, if $n = p$

(ii) BA is defined if number of columns in B = number of rows in A, $i.e.$, if $q = m$.

Let $A = [a_{ij}]_{m \times n}$, and $B = [b_{ij}]_{n \times p}$ be two matrices conformable for the product. AB, then AB is defined as the matrix $C = [c_{ij}]_{m \times p}$, where $c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj} =$

$\sum_{k=1}^{n} a_{ik}b_{kj}$ $i.e.$, $(i, j)^{th}$ element of AB = sum of the products of the elements of i^{th} row of A with the corresponding element of j^{th} column of B.

The rule for multiplication of two conformable matrices is called **row- by column method**.

Consider $\quad A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ and $B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix}$

Orders of A and B are 3×2 respectively. AB is defined and is of order 3×2.

$$A = C = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \\ c_{31} & c_{32} \end{bmatrix}$$

where c_{11} = sum of products of elements of 1^{st} row of A and 1^{st} column of B

$$= [a_{11}a_{12}a_{13}] \begin{bmatrix} b_{11} \\ b_{21} \\ b_{31} \end{bmatrix} = a_{11}b_{11} + a_{12}b_{21} + a_{13}a_{31}$$

c_{12} = sum of products of elements of 1^{st} row of A and 2^{st} column of B

$$= [a_{11}a_{12}a_{13}] \begin{bmatrix} b_{12} \\ b_{22} \\ b_{32} \end{bmatrix} = a_{11}b_{12} + a_{12}b_{22} + a_{13}a_{32}$$

c_{21} = sum of products of elements of 2^{nd} row of A and 1^{st} column of B

$$= [a_{21}a_{22}a_{23}] \begin{bmatrix} b_{11} \\ b_{21} \\ b_{31} \end{bmatrix} = a_{21}b_{11} + a_{22}b_{21} + a_{23}a_{31}$$

c_{22} = sum of products of elements of 2^{nd} row of A and 2^{nd} column of B

$$= [a_{21}a_{22}a_{33}] \begin{bmatrix} b_{12} \\ b_{22} \\ b_{32} \end{bmatrix} = a_{21}b_{12} + a_{22}b_{22} + a_{23}a_{32}$$

c_{31} = sum of products of elements of 3^{rd} row of A and 1^{st} column of B

$$= [a_{31}a_{32}a_{33}] \begin{bmatrix} b_{11} \\ b_{21} \\ b_{31} \end{bmatrix} = a_{31}b_{11} + a_{32}b_{21} + a_{33}a_{31}$$

c_{31} = sum of products of elements of 3^{rd} row of A and 2^{snd} column of B

$$= [a_{31}a_{32}a_{33}] \begin{bmatrix} b_{12} \\ b_{22} \\ b_{32} \end{bmatrix} = a_{31}b_{12} + a_{32}b_{22} + a_{33}a_{32}$$

Thus, $$AB = \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} & a_{11}b_{12} + a_{12}b_{22} + a_{13}b_{32} \\ a_{21}b_{11} + a_{22}b_{21} + a_{23}b_{31} & a_{21}b_{12} + a_{22}b_{32} + a_{23}b_{32} \\ a_{31}b_{11} + a_{32}b_{21} + a_{33}b_{12} & a_{31}b_{12} + a_{32}b_{22} + a_{33}b_{32} \end{bmatrix}$$

Example 2. $If\ A = \begin{bmatrix} 1 & -2 & 3 \\ 2 & 3 & -1 \\ -3 & 1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 1 & 2 & 0 \end{bmatrix}$, Obtain the products AB and B A and show that $AB \neq BA$.

Sol, $$AB = \begin{bmatrix} 1 & -2 & 3 \\ 2 & 3 & -1 \\ -3 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 1 & 2 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1(1)-2(0)+3(1) & 1(0)-2(1)+3(2) & 1(2)-2(2)+3(0) \\ 2(1)+3(0)-1(1) & 2(0)+3(1)-1(2) & 2(2)+3(2)-1(0) \\ -3(1)+1(0)+2(1) & 3(0)+1(1)+2(2) & -3(2)+1(2)+2(0) \end{bmatrix} = \begin{bmatrix} 4 & 4 & -2 \\ 1 & 1 & 10 \\ -1 & 5 & -4 \end{bmatrix}$$

$$BA = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 1 & 2 & 0 \end{bmatrix} \begin{bmatrix} 1 & -2 & 3 \\ 2 & 3 & -1 \\ -3 & 1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1(1)+0(2)+2(-3) & 1(-2)+0(3)+2(1) & 1(3)+0(-1)+2(2) \\ 0(1)+1(2)+2(-3) & 0(-2)+1(3)+2(1) & 0(3)+1(-1)+2(2) \\ 1(1)+2(2)+0(-3) & 1(-2)+2(3)+0(1) & 1(3)+2(-1)+0(2) \end{bmatrix} = \begin{bmatrix} -5 & 0 & 7 \\ -4 & 5 & 3 \\ 5 & 4 & 1 \end{bmatrix}$$

Order of AB and BA are the same but their corresponding elements are not equal.

Hence $AB \neq BA$.

PROPERTIES OF MATRIX MULTIPLICATION:

(i) Matrix multiplication is not commutative is general $.e.$, $AB \neq BA$.

(ii) Matrix multiplication is associative $i.e.$, $(AB)C = A(BC)$

(iii) Matrix multiplication is distributive with respect to matrix addition $i.e.$,

$$A(B+C) = AB + AC$$

(iv) If A and I are square matrices of the same order, then AI= IA =A.

(v) If A is a square matrix of order n, then $A \times A = A^2, A \times A \times A = A^3, A \times A \times A \ldots m$ Times $= A^m$ are all square matrices of order n.

Also, we define $A^0 = I$.

(vi) For any positive integer , $I^n = I$.

Example 3. Evaluate $A^2 - 3A + 9I, if I$ is the unit matrix of order 3 and $A = \begin{bmatrix} -6 & 1 & 2 \\ 5 & 4 & 4 \\ -3 & 1 & 2 \end{bmatrix}$.

Sol. $$A^2 = A \times A = \begin{bmatrix} 1 & -2 & 3 \\ 2 & 3 & -1 \\ -3 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & -2 & 3 \\ 2 & 3 & -1 \\ -3 & 1 & 2 \end{bmatrix} = \begin{bmatrix} -12 & -5 & 11 \\ 11 & 4 & 1 \\ -7 & 11 & -6 \end{bmatrix}$$

$$\therefore \quad A^2 - 3A + 9I_3 = \begin{bmatrix} -12 & -5 & 11 \\ 11 & 4 & 1 \\ -7 & 11 & -6 \end{bmatrix} - 3 \begin{bmatrix} 1 & -2 & 3 \\ 2 & 3 & -1 \\ -3 & 1 & 2 \end{bmatrix} + 9 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -12 & -5 & 11 \\ 11 & 4 & 1 \\ -7 & 11 & -6 \end{bmatrix} - \begin{bmatrix} 3 & -6 & 9 \\ 6 & 9 & -3 \\ -9 & 3 & 6 \end{bmatrix} + \begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix} = \begin{bmatrix} -6 & 1 & 2 \\ 5 & 4 & 4 \\ 2 & 8 & -3 \end{bmatrix}.$$

MATHEMATICAL INDUCTION:

Mathematical induction is a very useful device for proving results for all positive integers. If the result to be proved involves n, where n is a positive integer, then the proof by mathematical induction consists of the following:

Step1. Verify the result for $n = 1$.

Step 2.Assume the result to be true for $n = k$ and **then prove** that it is true for $n = k + 1$.

Now the result is true for $n = 1$

Using step 2, the result is true for $n = 1 + 1 = 2$

⇒The result is true for $n = 2 + 1 = 3$

⇒ The result is true for $n = 3 + 1 = 4$ and so on.

Hence, the result is true for all positive integers n.

Example 4. By mathematical induction, prove that if

$A = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}$, then $A^n = \begin{bmatrix} 1+2n & -4n \\ n & 1-2n \end{bmatrix}$ where n is any positive integer.

Sol. We prove the result by mathematical induction.

When $n = 1$, $\quad A^n = \begin{bmatrix} 1+2n & -4n \\ n & 1-2n \end{bmatrix}$

⇒ $\quad A^1 = \begin{bmatrix} 1+2.1 & -4.1 \\ 1 & 1-2.1 \end{bmatrix} = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix} = A$

⇒ The result is true when $n = 1$.

Let us assume that the result is true for any positive integer k.

$i.e.,$ let $\quad A^k = \begin{bmatrix} 1+2k & -4k \\ k & 1-2k \end{bmatrix}$ …(1)

Now, $\quad A^{k+1} = A^k.A = \begin{bmatrix} 1+2k & -4k \\ k & 1-2k \end{bmatrix}\begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}$ [Using (1)]

$$= \begin{bmatrix} 3(1+2k)-4k & -4(1+2k)+4k \\ 2k+1-2k & -4k-1+2k \end{bmatrix} = \begin{bmatrix} 3+2k & -4-4k \\ 1+k & -1-2k \end{bmatrix}$$

$$= \begin{bmatrix} 1+2(l+1) & 4(k+1) \\ k+1 & 1-2(k+1) \end{bmatrix}$$

⇒ The result is true for $n = k + 1$

Hence, by mathematical induction, the result is true for all positive integers n.

TRANSPOSE OF A MATRIX:

Given a matrix A, then the matrix obtained from A by changing its rows into columns and columns into rows is called the transpose of A and is denoted by A' or A^t;.

For example, $if\ A = \begin{bmatrix} 1 & 0 & 2 & 5 \\ 2 & -1 & 3 & 7 \end{bmatrix}$, then $A' = \begin{bmatrix} 1 & 2 \\ 0 & -1 \\ 2 & 3 \\ 5 & 7 \end{bmatrix}$

Clearly (i) if the order of A is $m \times n$ then order of A' is $n \times m$.

(ii) $(i,j)^{th}$ element of $A' = (j,i)^{th}$ element of A.

In symbols, $if\ A = [a_{ij}]_{m \times n}$, then $A' = [b_{ij}]_{n \times m}$, when $b_{ij} = a_{ij}$.

PROPERTIES OF TRANSPOSE OF A MATRIX

If A' and B' denote the transposes of A and B respectively, then

(i) $(A')' = A\ i.e.$, the transpose of the transpose of a matrix is the matrix itself.

(ii) $(a + b)' = A' + B' i.e.$, the transpose of the sum of two matrices is equal to the sum of their transposes.

(iii) $(AB)' = B'A'$ i.e., the transpose of the product of two matrices is equal to the product of their transposes taken in the reverse order.

SYMMETRIC MATRIX:

A square matrix $A = [a_{ij}]$ is said to be symmetric if $A' = A\ i.e.$, if the transpose of the matrix is equal to the matrix itself.

Thus, for a symmetric matrix $A = [a_{ij}], a_{ij} = a_{ji}$

For example, $\begin{bmatrix} -1 & 2 & 3 \\ 2 & 5 & 6 \\ 3 & 6 & 4 \end{bmatrix}, \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$ are symmetric matrices.

SKEW- SYMMETRIC MATRIX (OR SYMMETRIC MATRIX)

A square matrix $A = [a_{ij}]$ is said to be skew-symmetric if $A' = -A\ i.e.$, if the transpose of the matrix is equal to the negative of the matrix.

Thus, for a skew-symmetric matrix $A = [a_{ij}], a_{ij} = -a_{ji}$,

Putting $j = i,\ \ a_{ii} = -a_{ii} \Rightarrow 2a_{ii} = 0$ or $a_{ij} = 0$ for all i.

For example, $\begin{bmatrix} 0 & 2 & -3 \\ -2 & 0 & 1 \\ 3 & -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & h & -g \\ -h & 0 & f \\ g & -f & 0 \end{bmatrix}$ are skew- symmetric matrices

EVERY SQUARE MATRIX CAN UNIQUELY BE EXPRESSED AS THE SUM OF A SYMMETIC MATRIX AND A SKEW-SYMMETRIC MATRIX.

Let A be any square matrix.

Consider $B = \frac{1}{2}(A + A')$ and $C = \frac{1}{2}(A - A')$

$$B + C = \frac{1}{2}(A + A') + \frac{1}{2}(A - A') = A$$

Also $\quad B' = \frac{1}{2}(A + A')' = \frac{1}{2}[A' + (A')'] = \frac{1}{2}(A' + A) = B$

⇒ B is a symmetric matrix.

$$C' = \frac{1}{2}(A - A')' = \frac{1}{2}[A' - (A')'] = \frac{1}{2}[A' - A] = -\frac{1}{2}(A - A') = -C$$

⇒ C is a skew-symmetric matrix.

Hence every square matrix A can be expressed as $A = B + C,$ where $B = \frac{1}{2}(A + A')$ is a symmetric matrix and $C = \frac{1}{2}(A - A')$ is a skew-symmetric matrix.

To prove that there is only one way in which A can be expressed as the sum of a symmetric matrix and skew-symmetric matrix, suppose.

$$A = P + Q \qquad \ldots(1)$$

is another such representation in which P is a symmetric matrix, and Q is a skew-symmetric matrix.

$i.e.,$ $\quad P' = P$ and $Q' = -Q$

Then $\quad A' = (P + Q)' = P' + Q' = P - Q \qquad \ldots(2)$

From (1) and (2) $\quad A + A' = 2P\ and\ A - A' = 2Q$

∴ $\quad P == \frac{1}{2}(A - A') = B$ and $Q == \frac{1}{2}(A - A') = C$ so that $P + Q = B + C$

Hence the result.

ORTHOGONAL MATRIX

A square matrix A is called an orthogonal matrix $if\ AA' = A'A = I$ or $A' = A^{-1}$.

Note. If A and B are any two orthogonal matrices, AB will also be an orthogonal matrix.

NILOPTENT MATRIX

A square matrix A is said to be nilpotent if $A^2 = O \quad e.g.\ A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

It will be of index p if p is the least positive integer such that $A^p = O.$

IDEMPOTENT MATRIX

A square matrix A is idempotent if $A^2 = A$ $e.g.$ $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

A square matrix A is said to be idempotent of period n if n is the least positive integer such that $A^{n+1} = A$.

INVOLUTARY MATRIX

A square matrix A is said to be involuntary if $A^2 = I$ where I is a unit matrix.

$e.g.$ $$A = \begin{bmatrix} -5 & -8 & 0 \\ 3 & 5 & 0 \\ 1 & 2 & -1 \end{bmatrix}.$$

Example 5. If A and B are symmetric matrices, prove that AB-BA is a skew-symmetric matrix.

Sol. A and B are symmetric matrices.

$\Rightarrow$ $A' = A$ and $B' = B$ …(1)

Now, $(AB - BA)' = (AB)' - (BA)'$ $[\because (A-B)' = A' - B']$

$= B'A' - A'B'$ $[\because (AB)' = B'A']$

$= BA - AB$ $[Using(1)]$

$= -(AB - BA)$

$\therefore$ $(AB - BA)$ is a skew-symmetric matrix.

Example.6. Express the following matrix as the sum of a symmetric and a skew-symmetric matrix.

$$\begin{bmatrix} -1 & 7 & 1 \\ 2 & 3 & 4 \\ 5 & 0 & 5 \end{bmatrix}$$

Sol. Given matrix is $A = \begin{bmatrix} -1 & 7 & 1 \\ 2 & 3 & 4 \\ 5 & 0 & 5 \end{bmatrix}$ $\therefore$ $A' = \begin{bmatrix} -1 & 2 & 5 \\ 7 & 3 & 0 \\ 1 & 4 & 5 \end{bmatrix}$

Now, $$B = \frac{1}{2}(A + A') = \begin{bmatrix} -1 & 9/2 & 3 \\ 9/2 & 3 & 2 \\ 3 & 2 & 5 \end{bmatrix}$$

$$C = \frac{1}{2}(A - A') = \begin{bmatrix} 0 & 5/2 & -2 \\ -5/2 & 0 & 2 \\ 2 & -2 & 0 \end{bmatrix}$$

$$A = B + C = \begin{bmatrix} -1 & 9/2 & 3 \\ 9/2 & 3 & 2 \\ 3 & 2 & 5 \end{bmatrix} + \begin{bmatrix} 0 & 5/2 & -2 \\ -5/2 & 0 & 2 \\ 2 & -2 & 0 \end{bmatrix}$$

where B is symmetric and C is a skew-symmetric matrix.

ADJOINT OF A SQUARE MATRIX

The adjoint of a square matrix is the transpose of the matrix obtained by replacing each element of A by its co-factor in $|A|$. Adjoint of A is briefly written as adj. A.

Thus, $if A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$ then adj. A= transpose of $\begin{bmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{bmatrix} = \begin{bmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{bmatrix}$

where the capital letters denote the co-factors of corresponding small letters in

$$|A| = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$$

Note. If A is an n −rowed square matrix then

$$A(adj. A) = (adj. A)\, A = |A|\, I_n.$$

Properties of adjoint

(1) If A and B are two non-singular square matrices of the same order then

$$adj. (AB) = (adj. B)(adj. A).$$

(2) If A is a non singular matrix of order n then.

(i) $|adj. A| = |A|^{n-1}$ (ii) $adj. (adj. A) = |A|^{n-2} A$ (iii) $adj. A^T = (adj. A)^T$

INVERSE (OR RECIPROCAL) OF A SQUARE MATRIX

Let A be an n −rowed square matrix. If there exists an n −rowed square matrix B such that

$$AB = BA = I$$

then the matrix A is said to be invertible and B is called the inverse (or reciprocal) of A. The inverse of a square matrix if it exists is unique. The necessary and sufficient condition for a square matrix A to possess inverse is that A is non-singular $i.e.,\ |A| \neq 0.$

Note 1. Inverse of A is denoted by A^{-1}, thus $B = A^{-1}$ and $AA^{-1} = A^{-1}A = I.$

Note 2. $A^{-1} = \frac{adj. A}{|A|};\ |A| \neq 0.$ **Note 3**. $|A^{-1}| = |A|^{-1}.$

PROPERTIES OF INVERSE

(1) If A is invertible then so is A^{-1} and $(A^{-1})^{-1} = A$

(2) If A and B are two non-singular square matrices of the same order then

$$(AB)^{-1} = B^{-1}A^{-1}.$$

(3) If A is a non- singular square matrix then so is A' and $(A')^{-1} = (A^{-1})'$.

Example 8. $If\ A = \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$, (i) find A^{-1} (ii) Show that $A^3 = A^{-1}$

Sol. (i) $|A| = \begin{bmatrix} 3 & -3 & 4 \\ -2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix} = 1$ $\therefore A^{-1}$ exists

$$adj.A = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & -4 \\ -2 & 3 & -3 \end{bmatrix}$$

$\therefore$ $$A^{-1} = \frac{adjA.}{|A|} = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & -4 \\ -2 & 3 & -3 \end{bmatrix}$$

(ii) $$A^2 = A \times A = \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & -4 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -4 & 4 \\ 0 & -1 & 0 \\ -1 & 2 & -3 \end{bmatrix}$$

$$A^4 = A^2 \times A^2 = \begin{bmatrix} 3 & -4 & 4 \\ 0 & -1 & 0 \\ -1 & 2 & -3 \end{bmatrix}\begin{bmatrix} 3 & -4 & 4 \\ 0 & -1 & 0 \\ -1 & 2 & -3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

Now; $A.A^3 = A^3 = A = I \Rightarrow A^3 = A^{-1}$

Exercise

1. If A is a square matrix, prove that.

   (i) $A + A'$ is symmetric, (ii) $A - A'$ is skew-symmetric.

2. Prove that

   (i) If A, B are symmetric, then so is $A + B$

   (ii) If $A,\ B$ are skew-symmetric, then so is $A + B$

   (iii) If A is a square matrix, then AA' and $A'A$ are both symmetric.

   (iv) If A is symmetric, then B' AB is symmetric.

3. If $A = \begin{bmatrix} 1 & 2 & 1 \\ a & 0 & 4 \\ 1 & 1 & 1 \end{bmatrix}$ and adj. (adj. A)=A, find a.

4. If $A = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 0 & 2 \\ 4 & 5 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix}$, verify that $(AB)' = B'A'$.

5. Prove that following matrices are orthogonal.

(i) $\frac{1}{3}\begin{bmatrix} -2 & 1 & 2 \\ 2 & 2 & 1 \\ 1 & -2 & 2 \end{bmatrix}$ (ii) $\frac{1}{3}\begin{bmatrix} -8 & 4 & 1 \\ 1 & 4 & -8 \\ 4 & 7 & 4 \end{bmatrix}$

6. Express each of the following matrices as the sum of a symmetric and a skew- symmetric matrix.

(i) $\begin{bmatrix} 4 & 2 & -3 \\ 1 & 3 & -6 \\ -5 & 0 & -7 \end{bmatrix}$ (ii) $\begin{bmatrix} a & a & b \\ c & b & b \\ c & a & c \end{bmatrix}$

Answers

3. 3

6. (i) $\begin{bmatrix} 4 & \frac{3}{2} & -4 \\ \frac{3}{2} & 3 & -3 \\ -4 & -3 & 7 \end{bmatrix} + \begin{bmatrix} 0 & \frac{1}{2} & 1 \\ -\frac{1}{2} & 0 & -3 \\ -1 & 3 & 0 \end{bmatrix}$

(ii) $\begin{bmatrix} a & \frac{1}{2}(a+c) & \frac{1}{2}(b+c) \\ \frac{1}{2}(a+c) & b & \frac{1}{2}(b+c) \\ \frac{1}{2}(b+c) & \frac{1}{2}(a+b) & C \end{bmatrix} + \begin{bmatrix} 0 & \frac{1}{2}(a-c) & \frac{1}{2}(b-c) \\ \frac{1}{2}(c-a) & 0 & \frac{1}{2}(b-a) \\ \frac{1}{2}(c-b) & \frac{1}{2}(a-b) & 0 \end{bmatrix}$