Chapter
1

Polynomial and Transcendental Equation

To discuss methods of solving polynomial and transcendental equation.

Polynomial: An expression of the form.

$$P_n(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + ... + a_{n-1}x + a_n$$

where $a_0, a_1, a_2 ....a_n$ are constants, n is positive integer, is called a polynomial in x of degree n provided $a_0 \neq 0$

Zero of a Polynomial:

Any value of x which makes $P_n(x) = 0$ is called the zero of polynomial $P_n(x)$. Every polynomial $P_n(x)$ of degree n has exactly n zero.

Geometrically, the zero of the polynomial $P_n(x)$ is the value of x, where the graph of $P_n(x)$ crosses the axis of x.

Polynomial Equation: Let $P_n(x)$ be a polynomial in x then $P_n(x) = 0$ is known as polynomial equation

Transcendental Equation:

If a function *f(x)* contains some other functions such as trigonometric, logarithmic, exponential then *f(x) =0* is called transcendental equation.

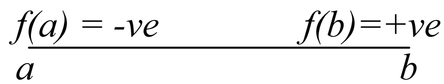
Method of solving equations:

The polynomial and transcendental equations can be solved by the following methods.

1. Bisection method.
2. Newton-Raphson Method.
3. False Position Method.

Bisection Method (Bolzano Method)

If $f(x)$ is continuous in the internal (a, b) such that $f(a)$ and $f(b)$ are of opposite sign such that $f(a).f(b)<0$ and the curve crosses the x-axis between a and b then the Ist approximation to the root is



$$x_1 = \frac{1}{2}(a+b)$$

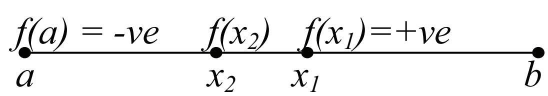
Now there are three cases, if

1. $f(x_1) = 0$ then x_1 is the root of $f(x)=0$
2. $f(x_1) < 0$ then the root lies between x_1 and b. [If $f(x_1) = -ve, f(b) = +ve$]
3. $f(x_1) > 0$ then the root lies between a and x_1

Suppose $f(x_1)>0$ then a < Root < x_1

Second Approximation to the root is

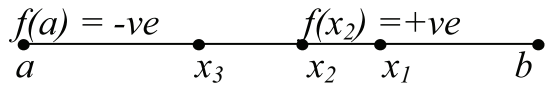
$$x_2 = \frac{1}{2}(a+x_1)$$



Again there are three cases, if

1. $f(x_2) = 0$ then x_2 is the root of $f(x)=0$
2. $f(x_2) < 0$ then root lies between x_2 and x_1.
3. $f(x_2) > 0$ then root lies between a and x_2

Suppose $f(x_2)>0$



Third Approximation

$$x_2 = \frac{1}{2}(a + x_2)$$

If 1. $f(x_3)=0$, then x_3 is the root of $f(x)=0$

2. $f(x_3)>0$, then the root lies between a and x_3 and so on.

<u>Order of convergence of iterative methods</u>

Convergence of an iterative method is judged by the order at which the error between successive approximations to the root decreases.

An iterative method is said to be k^{th} order convergent if k is the largest positive real number such that

$$\lim_{i \longrightarrow \infty} \left| \frac{e_i + 1}{e_i k} \right| \le A$$

where A is a non-zero finite number called asymptotic error constant and it depends on derivative of $f(x)$ at an approximate root x_i-e_i and e_{i+1} are the errors in successive approximations.

k^{th} order convergence gives us idea that in each iteration, the number of significant digits in each approximation increases k times.

The error in any step is proportional to the k^{th} power of the error in the previous step.

Order of convergence of bisection method.

In bisection method, the original interval is divided into half interval in each iteration. If we take mid points of successive intervals to be the approximations of the root, one half of the current interval is the upper bound to the error.

In bisection method, $e_{i+1} = 0.5\, e_i \; or \; \frac{e_{i+1}}{e_i} = 0.5$ .... (1)

where e_i and e_{i+1} are the errors in the i^{th} and $(i+1)^{th}$ iteration respectively.

Comparing (1) with $\lim_{i \longrightarrow \infty} \left| \frac{e_{i+1}}{e_i^k} \right| \le A$, we get k = 1 and A = 0.5

Thus the bisection method is Ist order convergent or linearly convergent.

In this method, the error decreases linearly with each step by a factor of 0.5. The convergence is slow. To achieve desired accuracy, a large number of iterations are required.

Problem: Perform five iterations of the bisection method to obtain the smallest positive root of the equation

$$f(x) = x^3 - 5x + 1 = 0$$

Solution: We have $f(x) = x^3 - 5x + 1 = 0$

$$f(0) = 1, \; f(1) = 1 - 5 + 1 = -3$$

$$and\; f(0).f(1) < 0$$

The Ist approximate root lies between 0 and 1.

$$x_1 = \frac{0+1}{2} = 1/2$$

$$f(1/2) = \frac{1}{8} - \frac{5}{2} + 1 = -1.375$$

$$as\; f(0)\, f\left(\frac{1}{2}\right) < 0$$

The second approximate lies between 0 and ½.

$$x_2 = \frac{0+\frac{1}{2}}{2} = \frac{1}{4}$$

$$f\left(\frac{1}{4}\right) = \frac{1}{64} - \frac{5}{4} + 1 = -0.234375$$

$$as\ f(0)\,f\left(\frac{1}{4}\right) < 0$$

The third approximate root lies between 0 and ¼ .

$$x_3 = \frac{0+\frac{1}{4}}{2} = \frac{1}{8}$$

$$f\left(\frac{1}{8}\right) = \frac{1}{\sqrt{12}} - \frac{5}{8} + 1 = +0.37695$$

$$as\ f\left(\frac{1}{8}\right) f\left(\frac{1}{4}\right) < 0$$

The fourth approximate root lies between $\frac{1}{8}$ and ¼ .

$$x_4 = \frac{\frac{1}{8}+\frac{1}{4}}{2} = \frac{3}{16}$$

$$f\left(\frac{3}{16}\right) = \left(\frac{3}{16}\right)^3 - \frac{15}{16} + 1 = +0.06909$$

$$as\ f\left(\frac{3}{16}\right).f\left(\frac{1}{4}\right) < 0$$

then $\frac{3}{16} < root < \frac{1}{4}$

$$Hence\ x_5 = \frac{\frac{3}{16}+\frac{1}{4}}{2} = \frac{7}{32}$$

$$Root = \frac{7}{32}$$

Problem: Solve $x^3 - 9x + 1 = 0$ for the root between $x=2$ and $x=4$ by Bolzano method.

Convergence of sequence:

A sequence $\sum x_n$ of successive approximations of a root $x=\alpha$ of the equation $f(x) = 0$ is said to converge to $x = \alpha$ with order $p \geq 1$ if and only if

$$|x_{n+1} - \alpha| \leq C|x_n - \alpha|^p, \; n \geq 0$$

c being some constant greater than zero.
If $p = 1$ and $0<C<1$, then convergence is called linear or Ist order. Constant C is called rate of linear convergence. Convergence is rapid or slow according as C is near 0 or 1.

Prove that Bisection method always converges.

Let $[p_n, q_n]$ be the interval at n^{th} step of bisection, having a root of the equation $f(x) =0$. Let x_n be the n^{th} approximation for the root. Then initially $p_1=a$ and $q_1=b$.

$$\Rightarrow x_1 = \text{Ist approximation} = \frac{p_1 + q_1}{2}$$

$$\Rightarrow p_1 < x_1 < q_1$$

Now either the root lies in $[a_1\ x_1]$ or in $[x_1, b]$

so either $[p_2, q_2] = [p_1, x_1]$ or $[p_2, q_2] = [x_1, q_1]$

$\Rightarrow$ Either $p_2=p_1$, $q_2=x_1$ or $p_2 = x_1$, $q_2 = q_1$

$\Rightarrow p_1 \leq p_2,\ q_2 \leq q_1$

Also $x_2 = \dfrac{p_2 + q_2}{2}$ *so that* $p_2 < x_2 < q_2$

Continuing in this way, at n^{th} step

$$x_n = \frac{p_n + q_n}{2},\ p_n < x_n < q_n$$

and $p_1 \leq p_2 \leq \ldots\ldots.. \leq p_n$ and $q_1 \geq q_2 \geq \ldots\ldots\ldots\ldots \geq q_n$

so $\{p_1, p_2, p_3 \ldots\ldots.. p_n \ldots\ldots.\}$ is bounded non decreasing sequence bounded by b and $\{q_1, q_2, q_3 \ldots\ldots.. q_n \ldots\ldots.\}$ is a bounded non-increasing sequence of numbers bounded by a. Hence both these sequence converges.

Let $\lim\limits_{n\longrightarrow\infty} p_n = p$ and $\lim\limits_{n\longrightarrow\infty} q_n = q$

Now since length of the internal is decreasing at every step, we get that $\lim\limits_{n\longrightarrow\infty}(q_n - p_n) = 0 \Rightarrow q = p$

Also $p_n < x_n < q_n$

$\lim p_n \le \lim x_n \le \lim q_n$

$p \le \lim x_n \le q$

$\Rightarrow \lim x_n = p = q$ .... (1)

Further since a root lies in $[p_n, q_n]$, we have

$$f(p_n).f(q_n) < 0$$

$$\lim_{n\longrightarrow\infty}[f(p_n).f(q_n)] \le 0$$

$$\Rightarrow f(p).f(q) \le 0$$

$$\Rightarrow [f(p)]^2 \le 0$$

But $f(p)^2 \ge 0$ being a square, so we get $f(p) = 0$, so p is a root of $f(x) = 0$................. (2)

So $\{x_n\}$ converges necessarily to a root of equation $f(x) = 0$.

This method is not rapidly converging but it is useful in the sense that it converges surely.

Newton-Raphson Method

Let x_0 be an approximate root of $f(x) = 0$ and let $x_1 = x_0 + h$ be the correct root so that $f(x_0+n) = 0$.

To find, h, we expand $f(x_0+h)$ by Taylor's Series.

$$f(x_0+h) = f(x_0) + hf'(x_0) + \frac{h^2}{2!} f''(x_0) + ...[f(x_0+h) = 0]$$

$0 = f(x_0) + hf'(x_0)$ (Neglecting higher and second order derivatives)

$$h = -\frac{f(x_0)}{f'(x_0)}$$, But $x_1 = x_0 + h$ so $x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$

x_1 is better approximation then x_0.

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

x_2 is better approximation than x_1

Successive approximations are, x_3, x_4 .... x_{n+1}.

$$\boxed{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}}$$

which is Newton Raphson formula.

Working Rule:

Step-I: Choose two close numbers *b* and *c* such that *f (b)* and *f(c)* are of opposite sign, and then the root lies between *b* and *c*.

Step-II: Out of *f (b)* and *f(c)* choose which is nearer to zero. If *f (b)* is nearer to zero then *b* is an initial approximate root (x_0) of the given equation.

Step-III: Apply Newton-Raphson formula to find out better approximate root x_1 as

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

Step-IV: Repeat the process to get successive approximations and stop when root are equal.

Problem: Find the smallest positive root of the equation

$$x^3 - 2x + 0.5 = 0.$$

Solution: Here $f(x) = x^3 - 2x + 0.5$

$f(0) = 0.5$

$f(0.1) = 0.001 - 0.2 + 0.5$

$= (0.301)\ (+ve)$

$f(0.3) = 0.027 - 0.6 + 0.5$

$= -0.073\ (-ve)$

Hence the root lies between 0.1 and 0.3

$f(0.3)$ is near to zero.

So 0.3 is Ist approximation, we have better approximation as 0.3.

To calculate 2nd Approximation by Newton's Method, *find,* $f'(x) = 3x^2 - 2$

So, $f'(0.3) = 3\ (0.3)2-2 = -1.73$

Second approximate root, $x_2 = 0.3 - \dfrac{f(0.3)}{f'(0.3)}$

$$= 0.3 - \frac{-0.073}{-1.73}$$

$$= 0.3 - \frac{73}{1730}$$

$$= 0.3 - 0.0422$$

$$= 0.2578$$

Problem: Find the real root of following, correct to three decimal places using Newton-Raphson method

$x^3 - 2x - 5 = 0$ ---------- (1)

Solution: Let $x^3-2x-5=0 = f(x)$, *so,* $f(2) = 8-4-5=-1\ (-ve)$ *and*

$f(2.5) = (2.5)^3-2(2.5)-5 = +5.625\ (+ve)$

Since $f(2)$ and $f(2.5)$ are of opposite sign, the root of (1) lies between 2 and 2.5, $f(2)$ is near to zero than $f(2.5)$ so 2 is better approximate root than 2.5

$$f'(x) = 3x^2 - 2 \Rightarrow f'(2) = 12 - 2 = 10$$

Let 2 be the approximate root of (1).
By Newton-Rophson Method

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 2 - \frac{f(2)}{f'(2)} = 2 - \frac{(-1)}{10} = 2.1$$

$$f(2.1) = (2.1)^3 - 2(2.1) - 5 = 9.261 - 4.2 - 5 \quad = 0.061$$

$$f'(2.1) = 3(2.1)^2 - 2 = 11.23$$

$$x_2 = 2.1 - \frac{f(2.1)}{f'(2.1)} = 2.1 - \frac{0.061}{11.23}$$

$$= 2.1 - 0.00543 = 2.09457$$

$$f(2.09457) = (2.09457)^3 - 2(2.09457) - 5$$

$$= 9.1893 - 4.18914 - 5 = 0.00016$$

$$f'(2.09457) = 3(2.09457)^2 - 2$$

$$= 13.16167 - 2 = 11.16167$$

$$x_3 = 2.09457 - \frac{f(2.09457)}{f'(2.09457)}$$

$$= 2.09457 - \frac{0.00016}{11.16167}$$

$$= 20.9457 - 0.000014$$

$$= 2.09456$$

As $x_3 = x_2$ correct up to four places of decimal, hence the root of the given equation is 2.0945, correct up to four places of decimal.

Rate of Convergence of Newton-Raphson Method:

Let x_n (approximate root) differs from the actual root α by a small quantity h_n so

$$x_n = \alpha + h_n \text{ .... (1)}$$

$$x_{n+1} = \alpha + h_{n+1} \quad \text{..................(2)}$$

By Newton Raphson Formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \quad \text{..................(3)}$$

Putting the values of x_{n+1} and x_n from (1) and (2) in (3),

we get $\alpha + h_{n+1} = \alpha + h_n - \dfrac{f(\alpha + h_n)}{f'(\alpha + h_n)}$

or $h_{n+1} = h_n - \dfrac{f(\alpha + h_n)}{f'(\alpha + h_n)}$

On expanding $f(\alpha + h_n)$ and $f'(\alpha + h_n)$ by Taylor's Series, we get

$$h_{n+1} = h_n - \frac{f(\alpha) + h_n f'(\alpha) + \frac{1}{2!}h^2{}_n \times f''(\alpha) + ...}{f'(\alpha) + h_n f''(\alpha) + ...}$$

Since $f(\alpha) = 0$, so

$$h_{n+1} = h_n - \frac{h_n f'(\alpha) + \frac{1}{2}h^2{}_n f''(\alpha) + ...}{f'(\alpha) + h_n f''(\alpha) + ...}$$

$$= \frac{h_n f'(\alpha) + h_n^{\,2} f''(\alpha) + ... - h_n f'(\alpha) - \frac{1}{2}h^2{}_n f''(\alpha) + ...}{f'(\alpha) + h_n f''(\alpha) + ...}$$

$$= \frac{\frac{1}{2}h_n^{\,2} f''(\alpha)}{f'(\alpha) + h_n f''(\alpha)}$$

$$h_{n+1} = h_n^{\,2}\left(\frac{f''(\alpha)}{2f'(\alpha)}\right) \quad [\textit{neglecting } f''(\alpha)]$$

so $h_{n+1} \propto h^2 n$ $\left[\dfrac{1}{2}\dfrac{f''(\alpha)}{f'(\alpha)} \text{ is constant}\right]$,

so convergence is of quadratic order *k=2*

Order of convergence, If $h_{n+1} = h_n^{\,k}$ where $x_n = \alpha + h_n$

$x_{n+1} = \alpha + h_{n+1}$

then *k* is called order of convergence

k=1, convergence is linear

k=2, convergence is quadratic.

Regular-Falsi Method or False Position Method → Oldest Method.

Let the root lie between a and b.These numbers a and b should be as close together as possible. Since the root lies between a and b, the graph of $y = f(x)$ must cross the *x-axis* between $x=a$ and $x=b$ and $f(a)$ and $f(b)$ must have opposite sign.

Now since any portion of a smooth curve is practically straight line for a short distance, it is legitimate to assume that change in $f(x)$ is proportional to change in x over a short interval.

The method of false position is based on this principle, for it assume that the graph of $y=f(x)$ is a straight line between the points (x_1, y_1) and (x_2, y_2), these points being on opposite side of *x-axis*.

Rule: Let $f(x) = 0$. Let $y = f(x)$ be represented by the curve AB. The real root of (1) is OP. The false position of the curve AB is taken as the chord AB. The chord AB cuts the *x-axis* at Q. The approximate root of $f(x) = 0$ is OQ. By this method, we find OQ.

Let A $[a, f(a)]$ and B $[b, f(b)]$ be the extremities of the chord AB.

The equation of chord AB is

$$y - f(a) = \frac{f(b) - f(a)}{b - a}(x - a)$$

To find OQ put $y=0$, $$0 - f(a) = \frac{f(b) - f(a)}{b - a}(x - a)$$

$$x - a = \frac{-(b-a)f(a)}{f(b) - f(a)}$$

Or $$x = \frac{a\,f(b) - b\,f(a)}{f(b) - f(a)}.$$

Rate of convergence of false position method:

Let f(x) = 0.......................... (1)

and α is the root of (1)

Let x_n be the approximate value of α and h_n be the error of x_n at the n^{th} stage.

$$x_n = \alpha + h_n \; [\, h_{n+1} = \text{error at } (n+1)^{th} \text{ stage} \,]$$

$$x_{n+1} = \alpha + h_{n+1} \text{ and } x_{n+2} = \alpha + h_{n+2}$$

By false, position method,

$$x_{n+2} = \frac{x_n f(x_{n+1}) - x_{n+1} f(x_n)}{f(x_{n+1}) - f(x_n)}$$

Putting the values, x_n, x_{n+1} *and* x_{n+2}, we get

$$\alpha + h_{n+2} = \frac{(\alpha + h_n) f(\alpha + h_{n+1}) - (\alpha + h_{n+1}) f(\alpha + h_n)}{f(\alpha + h_{n+1}) - f(\alpha + \mathrm{h}_n)}$$

On removing the brackets and arranging the terms on R.H.S., we get

$$\alpha + h_{n+2} = \frac{\alpha\left[f(\alpha + h_{n+1}) - f(\alpha + h_n)\right] + h_n f(\alpha + h_{n+1}) - h_{n+1} f(\alpha + h_n)}{f(\alpha + h_{n+1}) - f(\alpha + \mathrm{h}_n)}$$

$$\alpha + h_{n+2} = \alpha + \frac{h_n f(\alpha + h_{n+1}) - h_{n+1} f(\alpha + h_n)}{f(\alpha + h_{n+1}) - f(\alpha + \mathrm{h}_n)}$$

$$h_{n+2} = \frac{h_n f(\alpha + h_{n+1}) - h_{n+1} f(\alpha + h_n)}{f(\alpha + h_{n+1}) - f(\alpha + \mathrm{h}_n)}$$

Expanding by Taylor's Series

$$= \frac{h_n\left[f(\alpha)+h_{n+1}f'(\alpha)+\frac{h_{n+1}^2}{2!}f''(\alpha)+.........\right]-h_{n+1}\left[f(\alpha)+h_nf'(\alpha)+\frac{hn^2}{2!}f''(\alpha)+.........\right]}{\left[f(\alpha)+h_{n+1}f'(\alpha)+\frac{h_{n+1}^2}{2!}f''(\alpha)+.........\right]-\left[f(\alpha)+h_nf'(\alpha)+\frac{hn^2}{2!}f''(\alpha)+.........\right]}$$

$$h_{n+2} = \frac{h_n\left[0+h_{n+1}f'(\alpha)+\frac{h_{n+1}^2}{2!}f''(\alpha)+.........\right]-h_{n+1}\left[0+h_nf'(\alpha)+\frac{hn^2}{2!}f''(\alpha)+.........\right]}{\left[0+h_{n+1}f'(\alpha)+\frac{h_{n+1}^2}{2!}f''(\alpha)+.........\right]-\left[0+h_nf'(\alpha)+\frac{hn^2}{2!}f''(\alpha)+.........\right]}$$

$$= \frac{h_nh_{n+1}f'(\alpha)+h_n\frac{h_{n+1}^2}{2!}f''(\alpha)-h_{n+1}h_nf'(\alpha)-h_{n+1}\frac{hn^2}{2!}f''(\alpha)}{h_{n+1}f'(\alpha)+\frac{h_{n+1}^2}{2!}f''(\alpha)-h_nf'(\alpha)-\frac{hn^2}{2!}f''(\alpha)}$$

$$= \frac{\frac{h_nh^2{}_{n+1}}{2}f''(\alpha)-\frac{h_{n+1}h_n{}^2}{2}f''(\alpha)}{h_{n+1}f'(\alpha)-h_nf'(\alpha)} \text{(Ignoring higher powers of h)}$$

$$= \frac{\frac{h_nh^2{}_{n+1}}{2}\left[h_{n+1}-h_n\right]f''(\alpha)}{\left[h_{n+1}-h_n\right]f'(\alpha)} = h_nh_{n+1}\frac{f''(\alpha)}{2f'(\alpha)}$$

so $h_{n+2} = h_nh_{n+1} \times C$ where C = constant $= \frac{f''(\alpha)}{2f'(\alpha)}$

To find rate of convergence k such that

$h_{n+1} = A\ h_n^k$ ....(2)

[A = Constant of proportionality]

$$\Rightarrow h_{n+1} = A\ h_n^k \Rightarrow h_{n+1}^{\frac{1}{k}} = A^{\frac{1}{k}}h_n$$

$$\Rightarrow h_n = A^{-\frac{1}{k}}\ h^{\frac{1}{k}}{}_{n+1} \quad ....(3)$$

$$Again\ \ h_{n+2} = A\ h^k{}_{n+1} \quad ....(4)$$

Putting the values of h_n *and* h_{n+2} from (3) and (4) in (1), we get

$$Ah^K{}_{n+1} = A^{-\frac{1}{k}}\ h^{\frac{1}{K}}{}_{n+1}\left[h_{n+1} \times c\right]$$

$$Ah^K{}_{n+1} = A^{-\frac{1}{k}}\ h_{n+1}^{\frac{1}{K}+1}\text{(c)}$$

$$h^K{}_{n+1} = A^{\left(-\frac{1}{k}+1\right)}\ h_{n+1}^{\frac{1}{K}+1} \times c$$

Comparing the powers of h_{n+1} on both sides, we get.

$$k = \frac{1}{k} + 1 \Rightarrow k^2 - k - 1 = 0$$

$$\Rightarrow k = \frac{1 \pm \sqrt{1+4}}{2}$$

$$\Rightarrow k = \frac{1 \pm \sqrt{5}}{2} = \frac{1 + 2.2361}{2}\ \text{(Ignoring -ve sign)}$$

$$k = \frac{3.2361}{2} = 1.618$$

Putting the value of k in (2) we get

$$h_{n+1} = Ah_n^{1.618}$$

so the rate of convergence in false position method is 1.618

Problem: Find a positive root of $x^3 - 4x + 1 = 0$ by the method of false position.

Solution: Let $f(x) = x^3 - 4x + 1 = 0,\qquad f'(x) = 3x^2 - 4$

$f(0) = 1,\qquad f'(0) = -4$

$f(1) = 1-4+1 = -2$

Since $f(0)$ and $f(1)$ are of opposite sign, so the root lies between 0 and 1.

By false position method,

$$x = \frac{af(x) - bf(a)}{f(b) - f(a)}$$

$$x_1 = \frac{0f(1) - 1f(0)}{f(1) - f(0)} = \frac{-1}{-2-(1)} = \frac{1}{3}$$

$$f\left(\frac{1}{3}\right) = \left(\frac{1}{3}\right)^3 - 4\left(\frac{1}{3}\right) + 1 = \frac{1}{27} - \frac{4}{3} + 1 = -\frac{8}{27}$$

Since $f\left(\frac{1}{3}\right)$ and *f (0)* are of opposite sign, so the root lies between $\frac{1}{3}$ and 0.

$$x_2 = \frac{\frac{1}{3}f(0) - 0f\left(\frac{1}{3}\right)}{f(0) - f\left(\frac{1}{3}\right)} = \frac{\frac{1}{3}\times 1 - 0}{1 - \left(\frac{-8}{27}\right)} = \frac{9}{35}$$

$$f\left(\frac{9}{35}\right) = \left(\frac{9}{35}\right)^3 - 4\left(\frac{9}{35}\right) + 1 = \frac{729}{42875} - \frac{36}{35} + 1 = -\frac{496}{42875}$$

Since $f\left(\frac{9}{35}\right)$ and *f (0)* are of opposite sign, so the root lies between $\frac{9}{35}$ and 0.

$$x_3 = \frac{\frac{9}{35}f(0) - 0f\left(\frac{9}{35}\right)}{f(0) - f\left(\frac{9}{35}\right)} = \frac{\frac{9}{35}\times 1 - 0}{1 - \left(\frac{-496}{42875}\right)} = \frac{1225}{4819}$$

$$\sin ce\ x_2 = \frac{9}{35} = 0.2571 \quad \text{and} \quad x_3 = \frac{1225}{4819} = 0.2542$$

correct up to two decimal places. So the root of given equation is 0.25.

Problem: Using Regular false method find a positive root of the equation x^2*-2x-4=0* within an accuracy of 0.01.

Solution: $f(x) = x^2-2x-4=0, \qquad a=3,\ b=4$

$$f(3) = 9-6-4=-1$$
$$f(4) = 16-8-4=4$$

Since *f(3) and f(4)* are of opposite sign, the root of the given equation lies between 3 and 4.

By Regular False method.

$$x_1 = \frac{af(b)-bf(a)}{f(\text{b})-f(a)} = \frac{3f(4)-4f(3)}{f(4)-f(3)} = \frac{3\times4-4\times-1}{4-(-1)} = \frac{16}{5} = 3.2$$

$$f(3.2) = (3.2)^2 - 2(3.2) - 4 = -0.16$$

Since *f(3.2) and f(4)* are of opposite sign, so the root lies between 3.2 and 4.

$$x_2 = \frac{4f(3.2)-3.2f(4)}{f(3.2)-f(4)} = \frac{4(-0.16)-(3.2)(4)}{-0.16-4} = \frac{-13.44}{-4.16} = 3.230769$$

$$f(3.230769) = (3.230769)^2 - 2(3.230769) - 4 = -0.023670 \quad \text{(nearly zero)}$$

So approximate root = 3.230769

For Exact root $x^2-2x-4=0$

$$x = \frac{2\pm\sqrt{4+16}}{2} = 1\pm\sqrt{5}$$

Exact root = $1\pm\sqrt{5} = 1+2:23607 = 3.23607$

Error = Exact root - Approximate root

= [3.23607 - 3.230769] < 0.01

Exercise

1. Solve for a positive root of the equation $x^4 - x - 10 = 0$ using Newton – Raphson method.

2. Use Bisection method to find out the positive square root of 30 correct to 4 decimal places.

3. Find the root of the equation $xe^x = \cos x$ in the interval (0,1) using Regula-Falsi Method correct to the 4 decimal places.

4. Find a positive value of $(17)^{1/3}$ correct to four decimal places by Newton-Raphson Method.

5. Find the real root of the eq. $x \log_{10} x = 1.2$ by Regula Falsi Method correct to four decimal Places.

6. Use Bisection method to find the real root of $\cos x - xe^x = 0$ correct to 3 decimal places.

7. Solve for a real root of the equation $3x - \cos x - 1 = 0$ using Newton – Raphson method.

8. Find root between 0 and 1 of the equation $x^3 - 6x + 4 = 0$ using Newton – Raphson method.

9. Find the root of the equation $x^3 - 4x - 9 = 0$ using Regula-Falsi Method correct to the 3 decimal places.

10. Use Bisection method to find out the positive root of $x^3 - 3x + 1 = 0$ correct to 4 decimal places