Lecture -5
(Numerical Integration)

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- **Declaration :** — The study material provided in this lecture is original.

Chapter
5

Numerical Integration

Numerical integration is the process of evaluating a definite integral from the values of the function given in tabular form and when this process is applied to the integration of a function of single variable, and then the process is called quadrature.

The numerical integration of the tabulated value is solved by representing f(x) by an interpolation formula and then integrating it between the given intervals.

A General quadrature formula:

Let $I = \int_a^b f(x)dx$

Where *f(x)* is obtained by an interpolation formula in the interval (a, b) which is divided into n sub-intervals of equal width h. Let $x_0 = a,\ x_1 = a + h,\ x_2 = a + 2h$ .... $x_n = a + nh = b$ and corresponding values of *f(x)* be y_0, y_1, y_2 .... y_n.
The number of ordinates is (n+1).

If $y = f(x) = f(x_0 + ph)$ then $\frac{x - x_0}{h} = p \quad \Rightarrow \frac{dx}{h} = dp \Rightarrow dx = hdp$

By Newton's forward interpolation formula

$$y = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \ldots\ldots\ldots\ldots\ldots$$

Integrating both sides.

$$\int_{x_0}^{x_0+nh} ydx = \int_0^n \left[y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(\mathrm{p}-2)}{3!}\Delta^3 y_0 + \right.$$

$$+\frac{p(p-1)(\mathrm{p}-2)(\mathrm{p}-3)}{4!}\Delta^4 y_0 + \frac{p(p-1)(\mathrm{p}-2)(\mathrm{p}-3)(\mathrm{p}-4)}{5!}\Delta^5 y_0 +$$

$$\left. + \frac{p(p-1)(\mathrm{p}-2)(\mathrm{p}-3)(\mathrm{p}-4)(\mathrm{p}-5)}{6!}\Delta^6 y_0 + \ldots\ldots\ldots \right] hdp$$

$$= n\int_0^n \left[y_0 + p\Delta y_0 + \frac{p^2-p}{2!}\Delta^2 y_0 + \frac{p^3-3p^2+2p}{3!}\Delta^3 y_0 + \right.$$

$$+(p^4-6p^3+11p^2-6p)\frac{1}{24}\Delta^4 y_0 + (p^5-10p^4+35p^3-50p^2+24p)\frac{1}{120}\Delta^5 y_0 +$$

$$\left. +(p^6-15p^5+85p^4-225^3+274p^2-120p)\frac{1}{720}\Delta^6 y_0 + \ldots\ldots\ldots \right] dp \qquad [x_0=0]$$

On integrating with respect to p we get,

$$= h\left[py_0 + \frac{p^2}{2}\Delta y_0 + \left(\frac{p^3}{6} - \frac{p^2}{4}\right)\Delta^2 y_0 + \left(\frac{p^4}{4} - p^3 + p^2\right)\frac{\Delta^3 y_0}{6} \right.$$

$$+\left(\frac{p^5}{5} - \frac{3p^4}{2} + \frac{11p^3}{3} - 3p^2\right)\frac{\Delta^4 y_0}{24}$$

$$+\left(\frac{p^6}{6} - 2p^5 + \frac{35p^4}{4} - \frac{50p^3}{3} + 12p^2\right)\frac{\Delta^5 y_0}{120}$$

$$\left. +\left(\frac{p^7}{7} - \frac{15p^6}{6} + 17p^5 - \frac{225p^4}{4} + \frac{274p^3}{3} - 60p^2\right)\frac{\Delta^6 y_0}{720} + \ldots\ldots\ldots\ldots \right]_0^n$$

On Putting p = n, we get

$$= n\left[ny_0 + \frac{n^2}{2}\Delta y_0 + \left(\frac{n^3}{6} - \frac{n^2}{4}\right)\Delta^2 y_0 + \left(\frac{n^4}{4} - n^3 + n^2\right)\frac{\Delta^3 y_0}{6} \right.$$

$$+ \left(\frac{n^5}{5} - \frac{3n^4}{4} + \frac{11n^3}{3} - 3n^2\right)\frac{\Delta^4 y_0}{24}$$

$$+ \left(\frac{n^6}{6} - 2n^5 + \frac{35n^4}{4} - \frac{50n^3}{3} + 12n^2\right)\frac{\Delta^5 y_0}{120} \qquad (1)$$

$$\left. + \left(\frac{n^7}{6} - \frac{15n^6}{6} + 17n^5 - \frac{225n^4}{4} + \frac{274n^3}{3} - 60n^2\right)\frac{\Delta^6 y_0}{720} + \ldots\ldots\ldots\ldots \right]$$

This is Newton-cotes quadrature formula and from this we deduce different quadrature rules by taking n = 1, 2, 3, 6..................

Trapezoidal Rule:

We have to find the area of the region ABCD by trapezoidal Rule. Base AB is divided into n equal parts. First of all we find out the area of the Ist division and then area of the remaining divisions.

On taking n = 1, in the equation (1) the figures will be a trapezium and the curve will be a straight line passing through (x_0, y_0) and (x_1, y_1) and other integrals will be zero.

Putting n = 1 in (1)

$$\int_{x_0}^{x_0+h} f(x)dx = h\left[y_0 + \frac{1}{2}\Delta y_0 \right] = h\left[y_0 + \frac{1}{2}(y_1 - y_0) \right]$$

$$= h\left[\frac{y_0}{2} + \frac{y_1}{2} \right] = \frac{h}{2}[y_0 + y_1] \ldots\ldots\ldots(1)$$

Similarly

$$\int_{x_0+h}^{x_0+2h} f(x)dx = \frac{h}{2}\left[y_1 + y_2\right]$$

$$\int_{x_0+2h}^{x_0+3h} f(x)dx = \frac{h}{2}\left[y_2 + y_3\right]$$

....

....

$$\int_{x_0+(n-1)h}^{x_0+nh} f(x)dx = \frac{h}{2}\left[y_{n-1} + y_n\right]$$

On adding above n integrals, we get

$$\int_{x_0}^{x_0+h} f(x)dx + \int_{x_0+h}^{x_0+2h} f(x)dx + \int_{x_0+2h}^{x_0+3h} f(x)dx + ... + \int_{x_0+(n-1)h}^{x_0+nh} f(x)dx$$

$$= \frac{h}{2}\left[(y_0 + y_1) + (y_1 + y_2) + (y_2 + y_3) + ... + (y_{n-1} + y_n)\right]$$

$$= \frac{h}{2}\left[(y_0 + y_n) + 2(y_1 + y_2 + y_3 + ... + y_{n-1})\right]$$

So

$$\boxed{\int_{x_0}^{x_0+nh} f(x)dx = \frac{h}{2}\left[(y_0 + y_n) + 2(y_1 + y_2 + ... + y_{n-1})\right]}$$

Problem: Calculate an approximate value of $\int_0^{\frac{\pi}{2}} Sinx\, dx$ by Trapezoidal rule.

Solution: We divide the range $\left(0, \frac{\pi}{2}\right)$ into ten equal parts, so $h = \frac{\pi}{20}$

x	0	$\frac{\pi}{20}$	$\frac{\pi}{10}$	$\frac{3\pi}{20}$	$\frac{\pi}{5}$	$\frac{\pi}{4}$	$\frac{3\pi}{10}$
Sin x	0.0060	0.1564	0.3090	0.4540	0.5878	0.7071	0.8090
	y_0	y_1	y_2	y_3	y_4	y_5	y_6
x	$\frac{7\pi}{20}$	$\frac{2\pi}{5}$	$\frac{9\pi}{20}$	$\frac{\pi}{2}$			
Sin x	0.8910	0.9511	0.9877	1.0000			
	y_7	y_8	y_9	y_{10}			

Using Trapezoidal rule, approximate value is

$$\int_0^{\frac{\pi}{2}} Sinx\ dx = h\left\{\frac{1}{2}(y_0 + y_{10}) + (y_1 + y_2 + ... + y_9)\right\}$$

$$= \frac{\pi}{20}\left\{\frac{1}{2}(0+1) + 0.1564 + 0.3090 + 0.4540 + 0.5878\right.$$

$$\left. + 0.7071 + 0.8090 + 0.8910 + 0.9511 + 0.9877\ \right\}$$

$$= \frac{\pi}{20}(0.5 + 5.8531) = 0.9979 \qquad [\pi = 3.141592654]$$

Simpson's One Third Rule:

On putting n = 2 in (1), the curve through (x_0, y_0), (x_1, y_1) and (x_2, y_2) is a parabola (Polynomial of 2nd degree) and other integrals vanish.

So $$\int_{x_0}^{x_0+2h} f(x)dx = h\left[2y_0 + \frac{(2)^2}{2}\Delta y_0 + \left(\frac{2^3}{6} - \frac{2^2}{4}\right)\Delta^2 y_0\right]$$

$$= h\left[2y_0 + 2(y_1 - y_0) + \left(\frac{4}{3} - 1\right)(\Delta y_1 - \Delta y_0)\right]$$

$$= h\left[2y_0 + 2y_1 - 2y_0 + \frac{1}{3}\{y_2 - y_1 - (y_1 - y_0)\}\right]$$

$$= h\left[2y_0 + 2y_1 - 2y_0 + \frac{1}{3}y_2 - \frac{1}{3}y_1 + \frac{1}{3}y_0\right]$$

$$= h\left[\frac{1}{3}y_0 + \frac{4}{3}y_1 + \frac{1}{3}y_2\right]$$

So, $\int_{x_0}^{x_0+2h} f(x)\,dx = \frac{h}{3}[y_0 + 4y_1 + y_2]$

Similarly

$$\int_{x_0+2h}^{x_0+4h} f(x)\,dx = \frac{h}{3}[y_2 + 4y_3 + y_4]$$

$$\int_{x_0+6h}^{x_0+6h} f(x)\,dx = \frac{h}{3}[y_4 + 4y_5 + y_6]$$

....

....

$$\int_{x_0+(n-2)h}^{x_0+nh} f(x)\,dx = \frac{h}{3}[y_{n-2} + 4y_{n-1} + y_n]$$

Adding we get.

$$\int_{x_0}^{x_0+2h} f(x)dx + \int_{x_0+2h}^{x_0+4h} f(x)dx + \int_{x_0+6h}^{x_0+6h} f(x)dx + \ldots\ldots\ldots\ldots + \int_{x_0+(n-2)h}^{x_0+nh} f(x)dx$$

$$= \frac{h}{3}[(y_0 + 4y_1 + y_2) + (y_2 + 4y_3 + y_4) + (y_4 + 4y_5 + y_6) + \ldots\ldots\ldots\ldots + (y_{n-2} + 4y_{n-1} + y_n)]$$

So

$$\int_{x_0}^{x_0+nh} f(x)dx = \frac{h}{3}\left[(y_0 + y_n) + 2(y_2 + y_4 + y_6 + ... + y_{n-2}) + 4(y_1 + y_3 + y_5 + ... + y_{n-1})\right]$$

So $\int f(x)dx = \frac{h}{3}\left[(y_0 + y_n) + 2\sum y_e + 4\sum y_0\right]$

Geometric Proof:

Area $APQC = \int_{-n}^{+n} ydy = \int_{-n}^{n} (ax^2 + bx + c)dx$

$$= \left[\frac{ax^3}{3} + \frac{bx^2}{2} + cx\right]_{-n}^{n} = \frac{2h}{3}(ah^2 + 3c)$$

A (-h, y_0), B (0, y_1), C (h_1, y_2) lie on parabola.

$$y_0 = ah^2 - bh + c$$

$$y_1 = c$$

$$y_2 = ah^2 + bh + c$$

$$y_0 + y_2 = 2ah^2 + 2c = 2ah^2 + 2y_1$$

So, $\Rightarrow 2ah^2 = y_0 + y_2 - 2y_1$

$$\Rightarrow ah^2 = \frac{1}{2}(y_0 + y_2 - 2y_1)$$

So, APQC $= \frac{2h}{3}\left[\frac{1}{2}(y_0 + y_2 - 2y_1) + 3y_1\right]$

$$= \frac{h}{3}\left[y_0 + y_2 - 2y_1 + 6y_1\right]$$

$$= \frac{h}{3}\left[y_0 + 4y_1 + y_2\right]$$ so on

So, CQ RS $= \frac{h}{3}\left[y_2 + 4y_3 + y_4\right]$.........................

$$A = APQC + CQRS + ...$$

So, Area $= \frac{h}{3}\left[\times + 2E + 40\right]$

Problem: Calculate by Simpson's rule an approximate value of $\int_{-3}^{3} x^4 dx$ by taking seven equidistance intervals.

Solution: The seven equidistance intervals are -3, - 2, -1, 0 1, 2, 3 and the length of each interval is 1

x	-3	-2	-1	0	1	2	3
$Y = x^4$	81	16	1	0	1	16	81
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

So, $\int_{x_0}^{x_0+6h} y dx = \frac{h}{3}\left[(y_0 + y_6) + 2(y_2 + y_4) + 4(y_1 + y_3 + y_5)\ \right]$

$$\int_{x_0}^{3} x^4 dx = \frac{1}{3}\left[(81+81) + 2(1+1) + 4(16+0+16)\ \right]$$

$$= \frac{1}{3}\left[162 + 4 + 128\right] = \frac{1}{3} \times 294 = 98$$

So Approximate value of is $\int_{-3}^{3} x^4\ dx$ 98.

Simpson's Three eight Rule:

On putting n = 3 in (1), we consider three strips at a time. The curve passes through (x_0, y_0), (x_1, y_1), (x_2, y_2), (x_3, y_3). Other integrals vanish.

$$\int_{x_0}^{x_0+3h} f(x)dx$$

$$= h\left[3y_0 + \frac{3^2}{2}\Delta y_0 + \left(\frac{27}{6} - \frac{9}{4}\right)\Delta^2 y_0 + \left(\frac{81}{4} - 27 + 9\right)\frac{\Delta^3 y_0}{6}\right]$$

$$= h\left[3y_0 + \frac{9}{2}(y_1 - y_0) + \frac{9}{4}(\Delta y_1 - \Delta y_0) + \frac{3}{8}(\Delta^2 y_1 - \Delta^2 y_0)\right]$$

$$= h\left[3y_0 + \frac{9}{2}y_1 - \frac{9}{2}y_0 + \frac{9}{4}[y_2 - y_0 - (y_1 - y_0)] + \frac{3}{8}(\Delta y_2 - \Delta y_1 - \Delta y_1 + \Delta y_0)\right]$$

$$= h\left[\left\{3y_0 + \frac{9}{2}y_1 - \frac{9}{2}y_0 + \frac{9}{4}y_2 - \frac{9}{2}y_1 + \frac{9}{4}y_0\right\} + \frac{3}{8}\{(y_3 - y_2) - 2(y_2 - y_1) + (y_1 - y_0)\}\right]$$

$$= h\left[\frac{3}{4}y_0 + \frac{9}{4}y_2 + \frac{3}{8}(y_3 - y_2 - 2y_2 + 2y_1 + y_1 - y_0)\right]$$

$$= h\left[\frac{3}{4}y_0 + \frac{9}{4}y_2 + \frac{3}{8}y_3 - \frac{9}{8}y_2 + \frac{9}{8}y_1 - \frac{3}{8}y_0\right]$$

$$= h\left[\frac{3}{4}y_0 + \frac{9}{8}y_1 + \frac{9}{8}y_2 + \frac{3}{8}y_3\right]$$

So, $$\int_{x_0}^{x_0+3h} f(x)\,dx = \frac{3h}{8}[y_0 + 3y_1 + 3y_2 + 3y_3]$$

$$\int_{x_0+3h}^{x_0+6h} f(x)\,dx = \frac{3h}{8}[y_3 + 3y_4 + 3y_5 + 3y_6]$$

$$\int_{x_0+6h}^{x_0+9h} f(x)\,dx = \frac{3h}{8}[y_6 + 3y_7 + 3y_8 + 3y_9]$$

....

....

$$\int_{x_0+(n-3)h}^{x_0+nh} f(x)\,dx = \frac{3h}{8}[y_{n-3} + 3y_{n-2} + 3y_{n-1} + 3y_n]$$

Adding we get

$$\int_{x_0}^{x_0+nh} f(x)\,dx = \frac{3h}{8}\Big[(y_0 + y_n) + 3(y_1 + y_2 + y_4 + y_5 + \ldots\ldots + y_{n-1})$$
$$+ 2(y_3 + y_6 + y_9 + \ldots\ldots + y_{n-3})\Big]$$

Problem: A river is 80 meters wide. The depth *d* (in meters) of the river at a distance x from the bank is given by

x	0	10	20	30	40	50	60	70	80
d	0	4	7	9	12	15	14	8	3
	y_0	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8

Find approximately area of cross section of the river using Simpson's $\frac{3}{8}$ Rule.

Solution:

$$A=\int_0^{80} f(x)dx=\frac{3h}{10}\left[(y_0+y_8)+(y_1+y_2+y_4+y_5+y_7)+2(y_3+y_6)\right]$$

$$=\frac{30}{8}(87)=701.25 \text{ sq. metres. approx}$$

Weddle's Rule:

On Putting n = 6 in (1), we get

$$\int_{x_0}^{x_0+6h} f(x)dx=h\left[6y_0+18\Delta y_0+27\Delta^2 y_0+24\Delta^3 y_0+\frac{123}{10}\Delta^4 y_0\right.$$

$$\left.+\frac{33}{10}\Delta^5 y_0+\frac{41}{140}\Delta^6 y_0\right]$$

Taking $\frac{41}{140}\Delta^6 y_0=\frac{3}{10}\Delta^6 y_0$ (approx)

$$\int_{x_0}^{x_0+6h} f(x)dx=6h\left[y_0+3\Delta y_0+\frac{9}{2}\Delta^2 y_0+4\Delta^3 y_0+\frac{123}{60}\Delta^4 y_0\right.$$

$$\left.+\frac{11}{20}\Delta^5 y_0+\frac{1}{20}\Delta^6 y_0\right]$$

Putting $\Delta y_0=y_1-y_0$

$$\Delta^2 y_0=\Delta y_1-\Delta y_0=y_2-y_1-y_1+y_0=y_2-2y_1+y_0$$

$$\begin{aligned}\Delta^3 y_0 &= \Delta(\Delta^2 y_0) = \Delta(y_2 - 2y_1 + y_0)\\ &= (y_2 - 2y_2 + y_1) - (y_2 - 2y_1 + y_0)\\ &= y_3 - 3y_2 + 3y_1 - y_0\end{aligned}$$

$$\begin{aligned}\Delta^4 y_0 &= \Delta(\Delta^3 y_0) = \Delta(y_3 - 3y_2 + 3y_1 - y_0)\\ &= (y_4 - 3y_3 + 3y_2 - y_1) - (y_3 - 3y_2 + 3y_1 - y_0)\\ &= y_4 - 4y_3 + 6y_2 - 4y_1 + y_0\end{aligned}$$

$$\begin{aligned}\Delta^5 y_0 &= \Delta(\Delta^4 y_0) = \Delta(y_4 - 4y_3 + 6y_2 - 4y_1 + y_0)\\ &= (y_5 - 4y_4 + 6y_3 - 4y_2 + y_1) - (y_4 - 4y_3 + 6y_2 - 4y_1 + y_0)\\ &= y_5 - 5y_4 + 10y_3 - 10y_2 + 5y_1 - y_0\end{aligned}$$

$$\begin{aligned}\Delta^6 y_0 &= \Delta(\Delta^5 y_0) = \Delta(y_5 - 5y_4 + 10y_3 - 10y_2 + 5y_1 - y_0)\\ &= (y_6 - 5y_5 + 10y_4 - 10y_3 + 3y_2 - y_1) - (y_5 - 5y_4 + 10y_3 - 10y_2 + 5y_1 - y_0)\\ &= y_6 - 6y_5 + 15y_4 - 20y_3 + 15y_2 - 6y_1 + y_0\end{aligned}$$

On putting these values, we get.

$$\begin{aligned}\int_{x_0}^{x_0+6h} f(x)dx = 6h\Big[& y_0 + 3(y_1 - y_0) + \frac{9}{2}(y_2 - 2y_1 + y_0)\\ &+ 4(y_3 - 3y_2 + 3y_1 - y_0) + \frac{123}{60}(y_4 - 4y_3 + 6y_2 - 4y_1 + y_0)\\ &+ \frac{11}{20}(y_5 - 5y_4 + 10y_3 - 10y_2 + 5y_1 - y_0)\\ &+ \frac{1}{20}(y_6 - 6y_5 + 15y_4 - 20y_3 + 15y_2 - 6y_1 + y_0)\end{aligned}$$

$$= 6h\left[\left(1-3+\frac{9}{2}-4+\frac{41}{20}-\frac{11}{20}+\frac{1}{10}\right)y_0+\left(3-9+12-\frac{41}{5}+\frac{11}{2}-\frac{3}{10}\right)y_1\right.$$

$$+\left(\frac{9}{2}-12+\frac{123}{10}-\frac{11}{2}+\frac{3}{4}\right)y_2+\left(4-\frac{41}{5}+\frac{11}{2}-1\right)y_3$$

$$\left.+\left(\frac{41}{20}-\frac{11}{4}+\frac{3}{4}\right)y_4+\left(\frac{11}{20}-\frac{3}{10}\right)y_5+\frac{1}{20}y_6\right]$$

$$= 6h\left[\frac{1}{20}y_0+\frac{1}{4}y_1+\frac{1}{20}y_2+\frac{3}{10}y_3+\frac{1}{20}y_4+\frac{1}{4}y_5+\frac{1}{20}y_6\right]$$

Similarly

$$\int_{x_0+6h}^{x_0+12h} f(x)dx = \frac{3h}{10}\left(y_6+5y_7+y_8+6y_9+y_{10}+5y_{11}+y_{12}\right)$$

....

....

$$\int_{x_0+(n-6)h}^{x_0+nh} f(x)dx = \frac{3h}{10}\left(y_{n-6}+5y_{n-5}+y_{n-4}+6y_{n-3}+y_{n-2}+5y_{n-1}+y_n\right)$$

Adding, we get

$$f(x)dx = \frac{3h}{10}\left(y_0+5y_1+y_2+6y_3+y_4+5y_5+2y_6+........\right)$$

Remark: (1) Interval AB is divided into multiple of 6 sub-divisions.

(2) Weddle's Rule is more accurate than Simpson's Rule.

Problem: Evaluate $\int_0^1 \frac{dx}{(1+x^2)}$ by using

(1) Simpson's $\frac{3}{8}$ Rule (2) Weddle's Rule.

And hence find value of π dividing range into 6 six equal parts.

Solution: Here $\frac{1-0}{6}=\frac{1}{6}$

x	0	$\frac{1}{6}$	$\frac{2}{6}$	$\frac{3}{6}$	$\frac{3}{6}$
$\frac{1}{1+x^2}$	1.000	$\frac{36}{37}=0.97297$	$\frac{36}{40}=0.97297$	$\frac{36}{45}=0.80000$	$\frac{36}{52}=0.69231$
	y_0	y_1	y_2	y_3	y_4

By Simpson's $\frac{3}{8}$ Rule.

$$\int y dx=\frac{3h}{8}\left[(y_0+y_n)+3(y_1+y_2+y_4+y_5)+2(y_3+y_6)\right]$$

$$=\frac{3\times\frac{1}{6}}{8}\begin{bmatrix}(1.000+0.5000)+3(0.97297+0.90000\\+0.69231+0.059016+2(0.80000)\end{bmatrix}$$

$$=\frac{1}{16}\left[1.50000+3(3.15544)+2(0.80000)\right]$$

$$=\frac{1}{16}\left[12.56632\right]=0.785395$$

Again $\int_0^1 \frac{dx}{(1+x^2)}=\left(\tan^{-1}x\right)_0^1=\frac{\pi}{4}$

So $\frac{\pi}{4}=0.785395\Rightarrow \pi=3.14158$

(2) By Weddle's Rule.

$$\int y dx=\frac{3h}{10}\left[y_0+5y_1+y_2+6y_3+y_4+5y_5+y_6\right]$$

$$=\frac{3\times\frac{1}{6}}{10}\begin{bmatrix}1.0000+5(0.97297)+0.90000+6(0.80000)\\+0.69231)+5(0.59016)+0.50000\end{bmatrix}$$

$$=\frac{1}{20}\left[15.70796\right]=0.785398$$

So $\frac{\pi}{4}=0.785398\Rightarrow \pi=3.141592(approx)$.

Problems: Find the distance between two stations from the following data consisting of the speeds *v(+)* of an electric train at various times *t* after leaving one station until it stops at the next station. Apply Simpson's Rule

(v) (miles/hr)	0	13	33	39.5	40	40	36	15	0
t (min.)	0	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0
	y_0	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8

Solution: Here h = 0.5

If s (miles) be the distance covered in t (min), then $\frac{ds}{dt} = v$

$$\Rightarrow [s]_{t=0}^{4} = \int_0^4 v dt = \frac{h}{3}[(y_0 + y_8) + 2(y_2 + y_4 + y_6) + 4(y_1 + y_3 + y_5 + y_7)]$$

$$= \frac{0.5}{3}\left[(0+0) + 2\left(\frac{33}{60} + \frac{40}{60} + \frac{36}{60}\right) + 4\left(\frac{13}{60} + \frac{39.5}{60} + \frac{40}{60} + \frac{15}{60}\right)\right]$$

$$= \frac{0.5}{3}\left[2 \times \left(\frac{109}{60}\right) + 4\left(\frac{107.5}{60}\right)\right] = 1.8 miles.$$

Problem: Calculate by Simpson's rule an approximate value of $\int_{-3}^{3} x^4 dx$ by taking seven equidistance intervals.

Solution: The seven equidistance intervals are -3, - 2, -1, 0 1, 2, 3 and the length of each interval is 1

x	-3	-2	-1	0	1	2	3
$Y = x^4$	81	16	1	0	1	16	81
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

So, $$\int_{x_0}^{x_0+6h} y\,dx = \frac{h}{3}\left[(y_0 + y_6) + 2(y_2 + y_4) + 4(y_1 + y_3 + y_5)\right]$$

$$\int_{x_0}^{3} x^4 dx = \frac{1}{3}\left[(81 + 81) + 2(1 + 1) + 4(16 + 0 + 16)\right]$$

$$= \frac{1}{3}\left[162 + 4 + 128\right] = \frac{1}{3} \times 294 = 98$$

So Approximate value of is $\int_{-3}^{3} x^4\ dx$ 98.

Exact value $= \int_{-3}^{3} x^4\ dx = \left(\frac{x^5}{5}\right)_{-3}^{3} = \frac{1}{5}[243 + 243] = \frac{1}{5}(486) = 97.2$

Exercise

1. The velocity υ of a particle at a distance S from a point on its path is given by the table below:

S (meter)	0	10	20	30	40	50	60
υ (m / sec)	47	58	64	65	61	52	38

   Estimate the time taken to travel 60 meters by Simpson's 1/3rd rule and Simpson's 3/8th rule.

2. Find $\int_0^6 \frac{e^x}{1+x} dx$ approximately using Simpson's 3/8th rule on Integration.

3. Estimate the error in Trapezoidal rule, Simpson's one-third rule and Simpson's three-eight rule.

Exercise

4. Find the value of $\int_1^2 \frac{dx}{x}$ by Simpson's rule and hence evaluate the value of $\log_e 2$.

5. Compute $\int_0^4 e^x dx$ by Simpson's 1/3 rule with 10 subdivisons.

6. Evaluate $\int_0^1 \frac{dx}{1+x}$ by using I) Trapezoidal Rule II)Simpson's 1/3 rule III) Simpson's 3/8 Rule.

7. Evaluate $\int_0^1 \frac{dx}{1+x^2}$ by using Simpson's 3/8 rule taking h=1/6. Hence evaluate the value of π.

8. Evaluate $\int_0^1 \sqrt{1+x^3}\,dx$ by using I) Trapezoidal Rule II) Simpson's 1/3 rule.

9. Find the value of $\int_1^2 \frac{dx}{x}$ by Simpson's rule and Trapezoidal Rule take h = 0.25 in the given range?

10. The table below shows the temperature f(t) as a function of time t

t	1	2	3	4	5	6	7
F(t)	81	75	80	83	78	70	60

Use Simpson's 1/3 rule method to estimate $\int_0^7 f(t)\,dt$.