SOLUTION OF A SYSTEM OF LINEAR EQUATIONS

Consider the system of equations $\left.\begin{aligned} a_1x + b_1y + c_1z = d_1 \\ a_2x + b_2y + c_2z = d_2 \\ a_3x + b_3y + c_3z = d_3 \end{aligned}\right\}$ (3 equations in 3 unknowns)

In matrix notation, these equations can be written as

$$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

or AX =B

Where $A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$ is called the co-efficient matrix , $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ is the column matrix of unknowns, $B = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$ is the column matrix of constants.

If $d_1 = d_2 = d_3 = 0$, then $B = O$ and the matrix equation $AX = B$ reduces to $AX = O$, Such a system of equations is called a system of homogeneous linear equations. If at least one of $d_1,\ d_2,\ d_3$ is non-zero, then $B \neq O.$

Such a system of equations is called a system of non-homogeneous linear equations.

Solving the matrix equation $AX = B$ means finding $X, i.e.$, finding a column matrix $\begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix}$ such that $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix}$, Then $x = \alpha,\ y = \beta,\ z = \gamma$.

The matrix equation $AX = B$ need not always have a solution. It may have no solution or a unique solution or an infinite number of solutions.

A system of equations having no solution is called an inconsistent system of equation.

A system of equations having one or more solution is called a consistent system of equations.

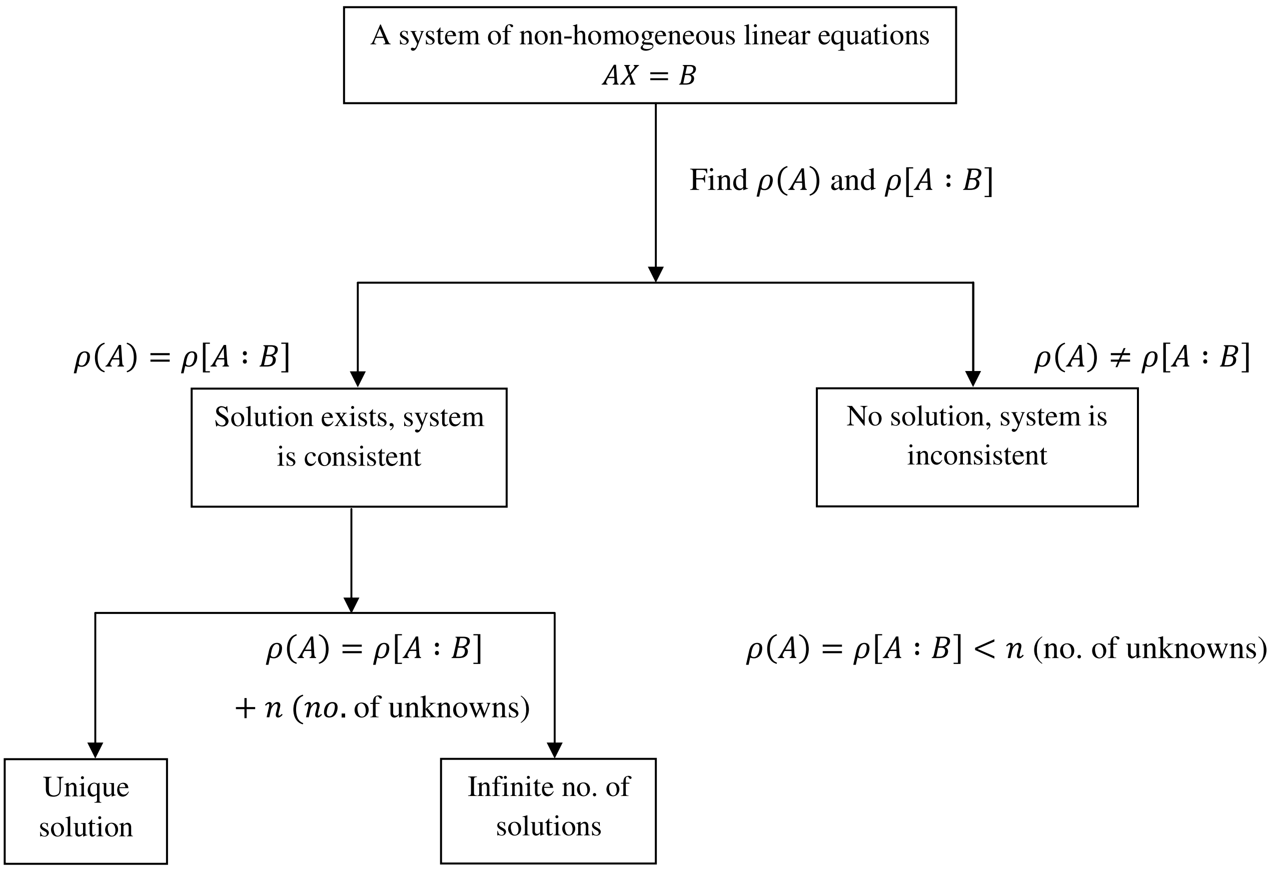
For a system of non-homogeneous linear equations $AX = B$.

(i) $if\ \rho[A : B] \neq \rho(A)$, the system is inconsistent.

(ii) $if\ \rho(A) = \rho(A) =$ number of unknowns, the system has a unique solution.

(iii) $\rho[A : B] = \rho(A) <$ number of unknowns, the system has an infinite number of solutions.

The matrix $[A : B]$ in which the elements of A and B are written side by side is called the augmented matrix.



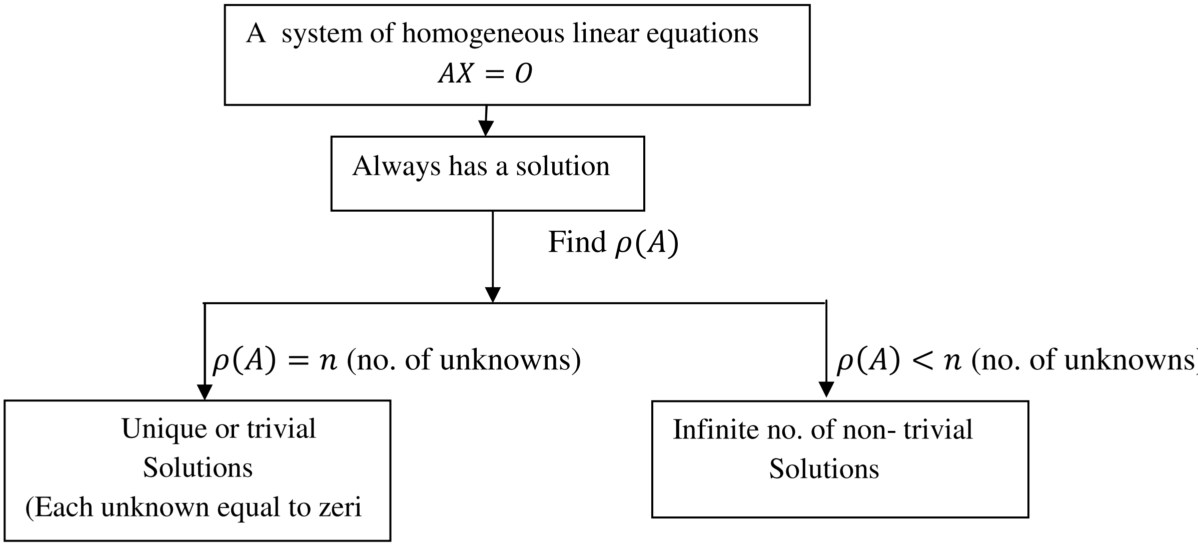
For a system of homogeneous linear equations AX = O

(i) $X = O$ is always a solution, This solution in which each unknown has the value zero is called the **Null Solution** or the **Trivial Solution.** Thus, a homogeneous system is always consistent.

A system of homogeneous linear equations has either the trivial solution or an infinite number of solutions.

(ii) $if\ \rho(A) =$ number of unknowns, the system has only the trivial solution.

(iii) $if\ \rho(A) <$ number of unknowns, the system has an infinite number of non-trivial solutions.



ILLUSTRATIVE EXAMPLES

Example 1. *Solve, with the help of matrices, the simultaneous equations:*

$$x + y + z = 3, x + 2y + 3z = 4, x + 4y + 9z = 6$$

Sol. Augmented matrix $[A : B] = \begin{bmatrix} 1 & 1 & 1 & \vdots & 3 \\ 1 & 2 & 3 & \vdots & 4 \\ 1 & 4 & 9 & \vdots & 6 \end{bmatrix}$

Operating $R_{21}(-1), R_{31}(-1)$

$$- \begin{bmatrix} 1 & 1 & 1 & \vdots & 3 \\ 0 & 1 & 2 & \vdots & 1 \\ 0 & 3 & 8 & \vdots & 3 \end{bmatrix}$$

Operating $R_{32}(-3)$

$$- \begin{bmatrix} 1 & 1 & 1 & \vdots & 3 \\ 0 & 1 & 2 & \vdots & 1 \\ 0 & 0 & 2 & \vdots & 0 \end{bmatrix}$$

$\therefore$ $\rho[A : B] = 3$. Also $\rho(A) = 3$.

Since, $\rho[A : B] = \rho(A) = 3$ (number of unknowns).

Hence the given system of equations is consistent and has a unique solution.

Equivalent system of equations is

$$x + y + z = 3$$

$$y + 2x = 1$$

$$2z = 0$$

$\Rightarrow$ $$x = 2, y = 1, z = 0$$

Example 2. *Solve the system of equations using matrix method:*

$$2x_1 + x_2 + 2x_3 + x_4 = 6, \qquad 6x_1 - 6x_2 + 6x_3 + 12x_4 = 36,$$

$$24 + 3x_2 + 3x_3 - 3x_4 = -1, \qquad 2x_1 + 2x_2 - x_3 + x_4 = 10,$$

Sol. Augmented matrix.

$$[A : B] = \begin{bmatrix} 2 & 1 & 2 & 1 & \vdots & 6 \\ 6 & -6 & 6 & 12 & \vdots & 36 \\ 4 & 3 & 3 & -3 & \vdots & -1 \\ 2 & 2 & -1 & 1 & \vdots & 10 \end{bmatrix}$$

Operating $R_{21}(-3), R_{31}(-2), R_{41}(-1)$

$$- \begin{bmatrix} 2 & 1 & 2 & 1 & \vdots & 6 \\ 0 & -9 & 0 & 9 & \vdots & 18 \\ 0 & 1 & -1 & -5 & \vdots & -13 \\ 0 & 1 & -.3 & 0 & \vdots & 4 \end{bmatrix}$$

Operating $R_2\left(-\frac{1}{9}\right)$

$$- \begin{bmatrix} 2 & 1 & 2 & 1 & \vdots & 6 \\ 0 & 1 & 0 & -1 & \vdots & -2 \\ 0 & 1 & -1 & -5 & \vdots & -1.3 \\ 0 & 1 & -3 & 0 & \vdots & 4 \end{bmatrix}$$

Operating $R_{32}(-1), R_{42}(-1)$,

$$- \begin{bmatrix} 2 & 1 & 2 & 1 & \vdots & 6 \\ 0 & 1 & 0 & -1 & \vdots & -2 \\ 0 & 0 & -1 & -4 & \vdots & -11 \\ 0 & 0 & -3 & 1 & \vdots & 6 \end{bmatrix}$$

Operating $R_{43}(-3)$

$$- \begin{bmatrix} 2 & 1 & 2 & 1 & \vdots & 6 \\ 1 & 1 & 0 & -1 & \vdots & -2 \\ 0 & 0 & -1 & -4 & \vdots & -11 \\ 0 & 0 & 0 & 13 & \vdots & 39 \end{bmatrix}$$

Which is echelon form

$\therefore$ $\rho[A : B] = 4$, Also $\rho(A) = 4$.

Since $\rho[A : B] = \rho(A) = 4$ (no. of variables)

Hence the system of equations is consistent and has a unique solution.

Equivalent system of equations is

$$2x_1 + x_2 + 2x_3 + x_4 = 6$$

$$x_2 - x_4 = -2$$

$$-x_3 - 4x_4 = -11$$

$$13x_4 = 39$$

On solving, we get $x_1 = 2, x_2 = 1, x_3 = -1, \quad x_4 = 3.$

Example 3. *Investigate, for what values of λ and μ do the system of equations*

$$x + y + z = 6, x + 2y + 3z = 10, x + 2y + \lambda z = \mu$$

have (i) no solution (ii) unique solution (iii) infinite solution?

Sol. Augmented matrix $[A : B] = \begin{bmatrix} 1 & 1 & 1 & \vdots & 6 \\ 1 & 2 & 3 & \vdots & 10 \\ 1 & 2 & \lambda & \vdots & \mu \end{bmatrix}$

Operating $R_{21}(-3), R_{31}(-2)$

$$- \begin{bmatrix} 1 & 1 & 1 & \vdots & 6 \\ 0 & 1 & 2 & \vdots & 4 \\ 0 & 1 & \lambda - 1 & \vdots & \mu - 6 \end{bmatrix}$$

Operating $R_{32}(-1)$

$$- \begin{bmatrix} 1 & 1 & 1 & \vdots & 6 \\ 0 & 1 & 2 & \vdots & 4 \\ 0 & 0 & \lambda - 3 & \vdots & \mu - 10 \end{bmatrix}$$

Case I. $If\ \lambda = 3, \mu \neq 10$

$$\rho(A) = 2, \rho[A : B] = 3$$

$\because \quad \rho(A) \neq \rho[A : B]$

∴ The system has **no solution,**

Case II. $If\ \lambda \neq 3, \mu$ may have any value

$$\rho(A) = \rho[A : B] = 3 = \text{number of unknowns}$$

∴ The system has unique solution.

CaseIII. $If\ \lambda = 3, \mu = 10$

$$\rho(A) = \rho[A : B] = 2 < \text{number of unknowns}$$

∴ The system has infinite number of solutions.

Example 4. *Test whether the following system of equations possess a non-trivial solution:*

$$x_1 + x_2 + 2x_3 + 3x_4 = 0$$

$$3x_1 + 4x_2 + 7x_3 + 10x_4 = 0$$

$$5x_1 + 7x_2 + 11x_3 + 17x_4 = 0$$

$$6x_1 + 8x_2 + 13x_3 + 16x_4 = 0$$

Sol. The given system is a homogeneous linear system of the form AX = O Coefficient matrix,

$$A = \begin{bmatrix} 1 & 1 & 2 & 3 \\ 3 & 4 & 7 & 10 \\ 5 & 7 & 11 & 17 \\ 6 & 8 & 13 & 16 \end{bmatrix}$$

Operating $R_{21}(-3), R_{31}(-5), R_{41}(-6)$

$$- \begin{bmatrix} 1 & 1 & 2 & 3 \\ 0 & 1 & 1 & 1 \\ 0 & 2 & 1 & 2 \\ 0 & 2 & 1 & -2 \end{bmatrix}$$

Operating $R_{23}(-2), R_{42}(-2)$

$$- \begin{bmatrix} 1 & 1 & 2 & 3 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & -1 \end{bmatrix}$$

Operating $R_{43}(-1)$

$$- \begin{bmatrix} 1 & 1 & 2 & 3 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & -4 \end{bmatrix}$$

$\therefore\ \rho(A) = 4$ (= no. of variables)

Hence the given homogeneous system of equations has trivial solution.

$\therefore$ $x_1 = 0,\ x_2 = 0,\ x_3 = 0$ and $x_4 = 0$

Example 5. *Show that the homogeneous system of equations.*

$$x + y\ \cos\gamma + z\cos\beta = 0$$

$$x\cos\gamma + y\ \cos\gamma + y + z\cos\alpha = 0$$

$$x\cos\beta + y\ \cos\alpha + z = 0$$

has non-trivial solution if $\alpha + \beta + \gamma = 0$

Sol. If the system has only non-trivial solutions, then

$$\begin{bmatrix} 1 & \cos\gamma & \cos\beta \\ \cos\gamma & 1 & \cos\alpha \\ \cos\beta & \cos\alpha & 1 \end{bmatrix} = 0$$

$$\Rightarrow \quad 1 - cos^2\,\alpha + \cos\gamma\,(\cos\alpha\cos\beta - \cos\gamma) + \cos\beta\,(\cos\gamma\cos\alpha - \cos\beta = 0)$$

$$\Rightarrow \quad sin^2\,\alpha - cos^2\;\beta - cos^2\;\gamma + 2\cos\alpha\cos\beta\cos\gamma = 0$$

$$\Rightarrow \quad -(cos^2\,\beta - sin^2\,\alpha) - cos^2\,\gamma + 2\cos\alpha\cos\beta\cos\gamma = 0$$

$$\Rightarrow \quad -\cos(\alpha+\beta)\cos\;(\alpha-\beta) - cos^2\,\gamma + 2\cos\alpha\cos\beta\cos\gamma = 0$$

$$|if\ \alpha+\beta+\gamma = 0$$

$$\Rightarrow \quad -\cos\;(-\gamma)\cos\,(\beta-\alpha) -\; cos^2\gamma + 2\cos\alpha\cos\beta\cos\gamma = 0$$

$$\Rightarrow \quad -\cos\gamma\,[\cos\,(\beta-\alpha) + \cos\,(\beta+\alpha)] + 2\cos\alpha\cos\;\beta\cos\;\gamma = 0$$

$$\Rightarrow \quad -2\cos\beta\cos\alpha\cos\gamma + 2\cos\alpha\cos\beta\cos\gamma = 0$$

Which is true.

Hence the given homogeneous system of equations has non- trivial solution if $\alpha + \beta + \gamma = 0$.

Example 6. *Show that the equations*

$$-2x + y + z = a$$

$$x - 2y + z = b$$

$$x + y = 2z = c$$

have no solution unless $a + b + c = 0$. *In which case they have infinitely many solutions? Find thses solution when* $a = 1, b = 1,\ c = -2$

Sol. Augmented matrix.

$$[A : B] = \begin{bmatrix} -2 & 1 & 1 & \vdots & a \\ 1 & -2 & 1 & \vdots & b \\ 1 & 1 & -2 & \vdots & c \end{bmatrix} \qquad |\rho(A) = 2$$

Operating R_{13}

$$- \begin{bmatrix} 1 & 1 & -2 & \vdots & c \\ 1 & -2 & 1 & \vdots & b \\ -2 & 1 & 1 & \vdots & a \end{bmatrix}$$

Operating $R_{21}(-1), R_{31}(2)$

$$- \begin{bmatrix} 1 & 1 & -2 & \vdots & a \\ 0 & -3 & 3 & \vdots & b-c \\ 0 & 3 & -3 & \vdots & a+2c \end{bmatrix}$$

Operating $R_{32}(1)$

$$- \begin{bmatrix} 1 & 1 & 1 & \vdots & c \\ 0 & -3 & 1 & \vdots & b-c \\ 0 & 0 & -2 & \vdots & a+b+c \end{bmatrix}$$

Case I. $if\ a+b+c \neq 2$

$\rho[A:B]-3 \neq \rho\ (A).$

Where A is the coefficient matrix.

Hence the system, inconsistent, have no solution.

Case II. If $a+b+c=0$

$$\rho[A:B]=2=\rho(A) \quad (<3)$$

Hence the system has infinite number of solution.

Equivalent system equations is

$$x+y-2z=-2 \qquad |Putting\ b+1,\ \ c=-2$$

$$-3y+3z=3$$

Let $z=k,\quad k$ being an arbitrary constant.

$y=k-1$

$x=k-1$

Hence the solutions are $x=k-1,\ \ y=k-1,\ \ z-k$

TEST YOUR KNOWLEDG

1. (i) Test the consistency of the following system of equations:

$$5x+3y+7z=4, 3x+26y+2z=9, 7x+2y+11z=5.$$

(ii) Test for the consistency of the following system of equations:

$$\begin{bmatrix} 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 7 \\ 5 & 6 & 7 & 8 \\ 10 & 11 & 12 & 13 \\ 15 & 16 & 17 & 18 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 7 \\ 8 \\ 9 \\ 14 \\ 19 \end{bmatrix}$$

(iii) Show that the equations $2x+6y+11=0, 6x+20y-6z+3=0$ and $6y-18z+1=0$ are not consistent.

2. Solve the following system of equations by matrix method:

(i) $x+y+z=8, x-y+2z=6, 3x+5y-7z=17$

(ii) $x+y+z=6, x-y+2z=5, 3x+y+z=8$

(iii) $x+2y+3z=1, 1x+3y+2z=2, 3x+3y+4z=1.$

3. (i) Test the consistency and hence solve the following set of equations:

$x_1 + 2x_2 + x_3 = 2, 3x_1 + x_2 - 2x_3 = 1, 4x_1 - 3x_2 - x_3 = 3, 2x_1 + 4x_2 + 2x_3 = 4$

(ii)Solve the system of linear equations using matrix method:

$$x + 2y + 3z = 5$$
$$7x + 11y + 13z = 17$$
$$19x + 23y + 29z = 31$$

(iii) Test for consistency and solve the following system of equations:

$$2x - y + 3z = 8$$
$$-x + 2y + z = 4$$
$$3x + y - 4z = 0$$

4. (i) Test for consistency, the equations $2x - 3y + 7z = 5, 3x + y - 3z = 13, 2x + 19y - 47z = 32$.

(ii) Verify that the following system of equations is inconsistent:

$$x + 2y + 2z = 1, 2x + y + z = -2, 3x + 2y + 2z = 3, y + z = 0.$$

(iii) Test for consistency of the equations:

$$2x - 3y + 7z = 5$$
$$3x + y - 3z = 13$$
$$2x + 19y - 47z = 32$$

(iv) Test the consistency and hence, solve the following set of equations:

$$10y + 3z = 0$$
$$3x + 3y + z = 1$$
$$2x - 3y - z = 5$$
$$x + 2y = 4$$

5. (i) Apply rank test to examine if the following system of equations is consistent, solve them.

$$2x + 4y - z = 9, 3x - y + 5z = 5, \; 8x + 2y + 9z = 19$$

(ii) Test the consistency for the following system of equations and if system, is consistent, solve them:

$$x + y + z = 6, x + 2y + 3z = 14, x + 4y + 7z = 30$$

6. Show that if $\lambda \neq -5$, the system of equations $3x - y + 4z = 3, x + 2y - 3z = -2, 3x + 5y + \lambda z = -3$ have a unique solution. if $\lambda = -5$, show that the equations are consistent, Determine the solutions in each case.

7. Foe what values of λ, the equations

$$x + y + z = 1, x + 2y + 4z = \lambda, x + 8y + 1z = \lambda^2$$

have a solution and solve them completely in each case.

8. (i) Verify that the following set of equations has a non-trivial solution:

$$x + 3y - 2z = 0, 2x - y + 4z = 0, x - 11y + 14z = 0.$$

(ii) Show that the following system of equations:

$x + 2y - 2u = 0, 2x - y - u = 0, x + 2z - u = 0, 4x - y + 3z - u = 0$ do not have a non-=trivial solution

9. Find the values of λ for which the equations.

$$x + (\lambda + 4)y + (4\lambda + 2)z = 0$$

$$x + 2(\lambda + 1)y + (3\lambda + 4)z = 0$$

$$2x + 3\lambda y + (3\lambda + 4)z = 0$$

have a non-trivial solution. Also find the solution in each case.

10. (i) Find the values of λ for which the equations

$$(11 - \lambda)z - 4y - 7z = 0$$

$$7x - (\lambda + 2)y - 5z = 0$$

$$10x - 4y - (6 + \lambda)z = 0$$

Possess a non-trivial solutions. For these values of λ, find the solution also.

(ii) For what values of λ the system of equations

$$2x - 2y + z = \lambda x, 2x - 3y + 2z = \lambda y, -x + 2y + 0z = \lambda z$$

Possess a non -trivial solution? Obtain its general solution.

ANSWERS

1. (i) Consistent (ii) Consistent with many solutions.

2. (i) $x = 5, y = \frac{5}{3}, z = \frac{4}{3}$ (ii) $x = 1, y = 2, z = 3$ (iii) $x = -\frac{3}{7}, y = \frac{8}{7}, z = -\frac{2}{7}$

3. (i) $x_1 = 1,\ x_2 = 0,\ x_3 = 1$ (ii) $x = -\frac{35}{18},\ y = \frac{2}{9}, z = \frac{13}{6}$ (iii) $x = 2, y = 2, z = 2$

4. (i) Inconsistent (ii) Inconsistent (iii) Inconsistent, no solutions exists

5. (i) $x = -\frac{19}{14}k + \frac{29}{14}, y = \frac{13}{14}k + \frac{17}{14}, z = k$ where k is arbitrary.

   (ii) $x = k - 2,\ y = 8 - 2k, z = k,$. where k is arbitrary

6. $\lambda \neq -5, x = \frac{4}{7},\ y = -\frac{9}{7}, z = 0;\ \lambda = -5, x = \frac{4-5k}{7}, y = -\frac{13k-9}{7}, z = k$ where k is arbitrary.

7. $\lambda = 1, 2$; when $\lambda = 1, x = 1 + 2k,\ y = -3k$ and $z = k$ when $\lambda = 2, x = 2k, y = 1 - 3k, z = k; k$ is arbitrary.

9. $\lambda = 2, x = 0, y = -5k, z = 3k;\ \lambda = -2, x = 4k, y = k, z = k.$

10. (i) $\lambda = 0, 1, 2$; when $\lambda = 0$, solution is (k, k, k)

    when $\lambda = 1$, solution is $(l, -k, 2k)$; wheb $\lambda = 2$, solutions is (2k, k, 2k).

    (ii) $\lambda = 1, -3$

    when $\lambda = 1, x = 2k_1 - k_2, y = k_1, z = k_2$; when $\lambda = -3, x = -k, y = -2k, z = k$.