VECTORS

Any ordered n −tuple of numbers is called an $n - vector.$ By an ordered n −tuple, we mean a set consisting of n numbers in which the place of each number is fixed. If $x_1,\ x_2,\ \ldots\ x_n$ be any n numbers then the ordered $n -$ tuple $X = (x_1, x_2,\ \ldots\ x_n)$ is called an $n -$ vector. Thus the coordinates of a point in space can be represented by a 3-vector (x, y, z) Similarly (1, 0, 2, -1) and (2, 7, 5, -3) are 4-vectors. The n numbers $x_1,\ x_{2,}\ \ldots\ x_n$ are called the components of the $n - vecotr\ \ X = (x_{1,} x_2,\ \ldots\ x_n)$. A vector may be written either as a $row\ vector$ or as a $column\ vector$. If A is a matrix of order $m \times n$, then each row of A will be an $n - vector$ and each column of A will be an $m - vector$. A vector whose components are all zero is called a zero vector and is denoted by O. Thus O= (0, 0, 0, … 0).

Let $X = (x_1, x_2, \ldots\ x_n)$ and $Y = (y_1,\ y_2,\ \ldots\ y_n)$ be two vectors.

Then $X = Y$ if and only if their corresponding components are equal.

$i.e.,$ if $x_i = y_i$, for $i = 1, 2\ \ldots n$

$$X + Y = (x_1 + y_1, x_2 + y_2, \ldots\ x_n + y_n)$$

If k be a scalar, then $kX = (kx_1, kx_2,\ \ldots\ kx_n)$

$n, n -$ vectors $e_1 = [1, 0, 0, \ldots 0], e_2[0, 1, 0, 0, \ldots, 0] \ldots, e_n = [0, 0, \ldots, 1]$ are called fundamental unit vectors or elementary vectors.

LINEAR DEPENDENCE AND LINEAR INDEPENDENCE OF VECTORS

A set of r n- vectors $X_{1,} X_2, \ldots, X_r$ is said to be $linearly\ \ dependen$ if there exit r scalars (numbers) $k_{1,}\ \ k_2\ \ \ldots, k_r$, not all zero, such that.

$$k_1X_1 + k_2X_2 + \ \ldots + k_rX_r = O$$

A set of $r\ n -$ vectors $X_1,\ \ X_{2,}\ \ \ldots\ ,\ X_r$, is said to be *linearly independent* if every relation of the type

$$k_1X_1 + k_2X_2 + \ \ldots + \ k_rX_r = O \text{ implies } k_1 = k_2 = \ \ldots = k_r = 0$$

To test the linear dependence of r given vectors, write them as row vectors. Add suitable multiples of one vector to the others so that the resulting $(r - 1)$ vectors have their first component zero. Choose anyone of these $(r - 1)$ vectors and add its multiples to the others so that the resulting $(r - 2)$ vectors have their second component zero. In this way continue, reducing the successive components to zero. If the final reduction gives a vector all of whose components are zero then the original vectors are linearly dependent. However, if the final reduction gives a vector all of whose components are not zero, then the original vectors are linearly independent.

Note I. If a set of vectors is linearly dependent, then at least one member of the set can be expressed as linear combination of the remaining vectors.

Note 2. The m, m −ddimensional vectors $X_{1,}\ X_2,\ X_3, \dots, X_m$ are linearly dependent if the rank of the matrix $[X_1,\ X_{2,}X_3, \dots, X_m]$ with the given vectors as columns is less than m.

Procedure to test linear dependence of vectors

Step 1. Construct coefficient matrix A with elements of given vectors as columns.

Step 2. Find $\rho(A)$.

Step 3. If $\rho(A) =$ no. of vectors then given set of vectors is linearly independent and if $\rho(A) <$ no. of vectors then given set of vectors is linearly dependent.

ANY NON-EMPTY SUBSET OF A LINEARLY INDEPENDENT SET OF VECTORS IS LINEARLY INDEPENDENT

Let $S_1 = (X_1,\ X_2,\ X_3, \dots, X_n)$

$S_2 = (X_1, X_2, X_3, \dots, X_r)$ where $r < n$

Clearly, S_2 is a subset of S_1,

Let $a_1X_1 + a_2X_2 + a_3X_3 + \dots + a_rX_r = O$ …(1)

$\Rightarrow$ $a_1X_1 + a_2X_2 + a_3X_3 + \dots + a_rX_r + 0, X_{r+1} + \cdots + 0, X_n = O$ … (2)

But S_1 is linearly independent

$\therefore$ From (2), $a_1 = a_2 = a_3 = \dots = a_r = 0$

Hence by eqn. (1), it is true only if all a's are zero

$\therefore$ Set S_2 is also linearly independent.

ANY SUPER SET OF A LINEARLY DEPENDENT SET OF VECTORS IS LINEARLY DEPENDENT

Let the set $(X_1,\ X_2, X_3, \dots X_r)$ be linearly dependent set, therefore $\exists\ r$ scalars $a_1, a_2, a_{3,} \dots a_r$ not all zero such that

$$a_1X_1 + a_2X_2 + a_2X_2 + \cdots + a_rX_r = O$$

$$\Rightarrow \quad a_1X_1 + a_2X_2 + a_2X_2 + \cdots + a_rX_r + 0,\ X_{r+1} + \dots + 0\ X_n = O$$

where all a's are not zero

Hence by definition, the set $[X_1,\ X_2,\ \dots, X_n]$ is linearly dependent.

ILLUSTRATIVE EXAMPLES

Example 1. $If\ X_1 = [1, 0, 0], X_2 = [0, 1, 0], X_3 = [0, 0, 1],\ find$

(i) $X_1 + X_2 + X_3$ (ii) $2X_1 + 3x_2 - X_{3.}$

Sol. (i) $X_1 + X_2 + X_3 = [1,0,0] + [0,1,0] + [0,0,1] = [1,1,1]$

(ii) $2X_2 + 3X_2 - X_3 = 2[1,0,0] + 3[0,1,0] - [0,0,1]$

$$= [2,0,0] + [0,3,0] - [0,0,1] = [2,3,-1].$$

Example 2. *Show that the vectors* $X_1 = [1,2,3], X_2 = [2,-2,0]$ *form a linearly independent set*

Sol. Coefficient matrix

$$A = \begin{bmatrix} 1 & 2 \\ 2 & -2 \\ 3 & 0 \end{bmatrix}$$

Portaging $R_{21}(-2), R_{31}(-3)$

$$- \begin{bmatrix} 1 & 2 \\ 0 & -6 \\ 0 & -6 \end{bmatrix}$$

Operating $R_{32}(-1)$

$$- \begin{bmatrix} 1 & 2 \\ 0 & -6 \\ 0 & 0 \end{bmatrix}$$

$\therefore$ $\rho(A) = 2$(=no. of vectors)

Hence the given set of vectors is linearly independent.

Example 3. *Shoe that the vectors* $X_1 = (1,2,4), X_2 = (2,-1,3), X_3 = (0,1,2)$ *and* $X_4 = (-3,7,2)$*are linearly dependent and find the relation between them*

Sol. Coefficient matrix

$$A = \begin{bmatrix} 1 & 2 & 0 & -3 \\ 2 & -1 & 1 & 7 \\ 4 & 3 & 2 & 2 \end{bmatrix}$$

Operating $R_{21}(-2), R_{31}(-4)$

$$- \begin{bmatrix} 1 & 2 & 0 & -3 \\ 0 & -5 & 1 & 13 \\ 0 & -5 & 2 & 14 \end{bmatrix}$$

Operating $R_{32}(-1)$

$$- \begin{bmatrix} 1 & 2 & 0 & -3 \\ 0 & -5 & 1 & 13 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$\therefore$ $\rho(A) = 3$ (< no. of vectors)

Hence the given set of vectors is linearly dependent.

Let the relation between the vectors be

$$a_1X_1 + a_2X_2 + a_3X_2 + a_4X_4 = O \qquad \ldots (1)$$

where a_1, a_2, a_3 and a_4 are scalars given by

$$a_1 + 2a_2 - 3a_4 = 0 \qquad \ldots (2)$$

$$-5a_2 + a_3 + 13a_4 = 0 \qquad \ldots (3)$$

$$a_3 + a_4 = 0 \qquad \ldots (4)$$

From (4), $\frac{a_3}{1} = \frac{a_4}{-1} = k$ (say)

so that, $a_3 = k$ and $a_4 = -k$

From (3), $-5a_2 + k - 13k = 0 \Rightarrow a_2 = -\frac{12}{5}k$

From (2), $a_1 - \frac{24}{5}k + 3k = 0 \Rightarrow a_1 = \frac{9}{5}k$

Substituting these values is (1), we get

$$\frac{9}{5}kX_1 - \frac{12}{6}kX_2 + kX_3 - kX_4 = O$$

$$\Rightarrow \quad 9X_1 - 12X_2 + 5X_3 - 5X_4 = O \qquad (\because k \neq 0)$$

Example 7. *Find the value of λ for which the vectors* (*1, -2, λ*), (*2, -1, 5*) *and* (*3, -5, 7λ*) *are linearly dependent.*

Sol. Coefficient matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ -2 & -1 & -5 \\ \lambda & 5 & 7\lambda \end{bmatrix}$$

The given vectors will be linearly dependent when $\rho(A) <$ no. of vectors ($i.e.$ 3) Now, $\rho(A) < 3$ is possible only if $|A| = 0$

Hence, $$|A| = \begin{bmatrix} 1 & 2 & 3 \\ -2 & -1 & -5 \\ \lambda & 5 & 7\lambda \end{bmatrix} = 0$$

$$\Rightarrow \quad 1(-7\lambda + 25) - 2(-14\lambda + 5\lambda) + 3(-10 + \lambda) = 0$$

$$\Rightarrow \quad 14\lambda = 5$$

$$\Rightarrow \quad \lambda = \frac{5}{14}$$

TEST YOUR KNOWLEDGE

1. For the vectors $X = [3, -9, 12]$, and $Y = [-4, 12, -16]$ show that $4X + 3Y = O$.
2. Show that the elementary unit vectors $E_1 = [1, 0, 0, \ldots, 0]$, $E_2 = [0, 1, 0, \ldots, 0], \ldots, E_n = [0, 0, 0, \ldots, 1]$ are linearly independent.

3. (i) Show that the column vectors of $A = \begin{bmatrix} 1 & 0 & 0 \\ 6 & 2 & 1 \\ 4 & 3 & 2 \end{bmatrix}$ are linearly independent.

(ii) Show that row vectors of the matrix $\begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$ are linearly independent.

4. Show that the vectors $X_1 = [a_1, b_1], X_2 = [a_2, b_2]$ are linearly dependent iff $a_1b_2 - a_2b_1 = 0$.

5. Prove that the four vectors $X_1 = [1, 0, 0], X_2 = [0, 1, 0], X_2 = [0, 0, 1], X_4 = [1, 1, 1]$ in $V_3(c)$ from a linearly dependent set but any three of them are linearly independent.

6. Show that the vectors $X_1 = [2, 3, 1, -1],\ X_2 = [2, 3, 1, -2], X_3 = [4, 6, 2, 1]$ are linearly dependent. Express one of the vectors as a linear combination of the others.

7. Examine the linear dependence or independence of the following set of vectors:

(i) $[3, 2, 4], [1, 0, 2], [1, -1, -1]$ (ii) $[2, 3, 1, -3], [2, 3, 1, -2], [4, 6, 2, -3]$

(iii) $[1, 1, 0], [3, 1, 3], [5, 3, 3]$ (iv) $[1, 2, 1, 2], [0,1, 2,1], [1, 4, 3, 2]$

8. Show using a matrix that the set of vectors $X = [1, 2, -3, 4], Y = [3, -1, 2, 1], Z = [1, -5, 8, -7]$ is linearly dependent. Determine a maximum subset of linearly independent vectors and express the others as a linear combination of these.

9. If the vectors $(0, 1, a), (1, a, 0)$ and $(a, 1, 0)$ are linearly dependent, then find the value of a.

10. Show that the vectors $x_1 = (1, -1, 1),\ x_2 = (2, 1, 1), x_3 = (3, 0, 2)$ are linearly dependent. Find the relation between them.

11. Examine the following vectors for linear dependence and find the relation between them, if possible:

$X_1 = (1, 1, -1, 1), X_2 = (1, -1, -2, -1),\ X_3 = (3, 1, 0, 1)$

ANSWERS

6. $5X_1 - 3X_2 - X_3 = O$

7. (i) linearly independent (ii) linearly dependent (iii) linearly dependent (iv) linearly independent

8. 2; $X_3 = -2X_1 + X_2$

9. $a = 0, \neq \sqrt{2}$

10. $x_1 + x_2 = x_3$

11. linearly dependent; $2X_1 + X_2 = X_3$.