# Department of Mathematics

## Faculty of Engineering &Technology

## V.B.S. Purvanchal University, Jaunpur

## Prepared by: Dr. Sushil Shukla


# Tutorial Sheet

**Topic: Module 2: Multivariable Calculus-II [08]**

Improper integrals, Beta & Gama function, and their properties, Dirichlet's integral and its applications, Application of definite integrals to evaluate surface areas and volume of revolutions.

Determine if each of the following integrals converge or diverge. If integral converges Determine its value

1) $\int_0^\infty (1+2x)e^{-x}dx$ ANS: CONVERGES, VALUE = 3

2) $\int_{-\infty}^0 (1+2x)e^{-x}dx$ ANS: DIVERGES

3) $\int_{-5}^1 \frac{1}{10+2x}dx$ ANS: DIVERGES

4) $\int_1^2 \frac{4x}{3\sqrt{x^2-4}}dx$ ANS: CONVERGES, VALUE = $(3)^{5/3}$

5) $\int_{-\infty}^1 \sqrt{6-x}\,dx$ ANS: DIVERGES

6) $\int_2^\infty \frac{9}{(1-3x)^4}dx$ ANS: CONVERGES, VALUE = 1/125

7) $\int_0^4 \frac{x}{x^2-9}\,dx$ ANS: DIVERGES

8) $\int_{-\infty}^{\infty} \frac{6x^3}{\left(x^4+1\right)^2}\,dx$ ANS: CONVERGES, VALUE = 0

9) $\int_1^4 \frac{1}{x^2+x-6}\,dx$ ANS: CONVERGES, VALUE = 1

10) $\int_{-\infty}^{0} \frac{e^{\frac{1}{x}}}{x^2}\,dx$ ANS: CONVERGES, VALUE = 1

11) $\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx$ ANS: CONVERGES, VALUE = $\pi$

12) $\int_0^{\infty} e^{-x}\,dx$ ANS: CONVERGES, VALUE = 1

13) $\int_0^2 \frac{2x}{x^2-4}\,dx$ ANS: DIVERGES

14) $\int_1^{\infty} \frac{\log x}{x^2}\,dx$ ANS: CONVERGES, VALUE = 1

15) $\int_0^1 \frac{1}{\sqrt{x}}\,dx$ ANS: CONVERGES, VALUE = 2

16) $\int_1^{\infty} \frac{1}{(3x+1)^2}\,dx$ ANS: CONVERGES, VALUE = 1/12

17) $\int_{-\infty}^{\infty} \cos(\pi t)\,dt$ ANS: DIVERGES

18) $\int_6^8 \frac{4}{(x-6)^3}\,dx$ ANS: DIVERGES

19) $\int_{-\infty}^{0} \frac{1}{2x-5}\,dx$ ANS: DIVERGES

20) $\int_0^\infty \frac{1}{x^2+3x+2}dx$ ANS: CONVERGES, VALUE = log (2)

21) $\int_0^1 \frac{1}{\sqrt{1-x^2}}dx$ ANS: CONVERGES, VALUE = $\frac{\pi}{2}$

22) $\int_0^\infty \frac{x}{(x^2+2)^2}dx$ ANS: CONVERGES, VALUE = $\frac{1}{4}$

23) $\int_{-\infty}^{-1} e^{-2t}\,dt$ ANS: DIVERGES

24) $\int_{-\infty}^{\infty} x^3 e^{-x^4}\,dx$ ANS: CONVERGES, VALUE = 0

25) $\int_0^\infty \frac{e^x}{e^{2x}+3}dx$ ANS: CONVERGES, VALUE = $\frac{\sqrt{3}\pi}{9}$

26) $\int_2^3 \frac{1}{\sqrt{3-x}}dx$ ANS: CONVERGES, VALUE = 2

27) Evaluate $\int_0^1 \frac{dx}{\sqrt{1-x^4}}$ ANS: $\frac{\sqrt{\pi}}{4}\frac{\Gamma(1/2)}{\Gamma(3/4)}$

28) Prove that $\Gamma\left(\frac{1}{2}\right)=\sqrt{\pi}$

29) $Given\,that \int_0^\infty \frac{x^{n-1}}{1+x}dx=\frac{\pi}{\sin n\pi}, then\,show\,that\ \Gamma(n)\Gamma(1-n)=\frac{\pi}{\sin(n\pi)}$

30) Show that

$$\overline{|m}\,\overline{\left|m+\frac{1}{2}\right.}=\frac{\sqrt{\pi}}{2^{2m-1}}\overline{|2m}$$ where $m$ is the integer.

Hence show that $B(m,n)=2^{1-2m}B\left(m,\frac{1}{2}\right)$

*31) Complete:*

a) $\Gamma(-1/2)$ ANS: $-2\sqrt{\pi}$

b) $\Gamma(-5/2)$ ANS: $-\frac{8}{15}\sqrt{\pi}$

32) Evaluate in term of Beta function.

a) $\int_0^1 \frac{x}{\sqrt{1-x^2}}dx$ ANS: $\frac{1}{2}\beta\left(1,\frac{1}{2}\right)$

b) $\int_0^3 \frac{dx}{\sqrt{9-x^2}}$ ANS: $\frac{1}{2}\beta\left(\frac{1}{2},\frac{1}{2}\right)$

c) $\int_0^1 \frac{x\,dx}{\sqrt{1-x^5}}$ ANS: $\frac{1}{5}\beta\left(\frac{2}{5},\frac{1}{2}\right)$

d) $\int_0^1 \frac{x^2\,dx}{\sqrt{1-x^5}}$ ANS: $\frac{1}{5}\beta\left(\frac{3}{5},\frac{1}{2}\right)$

33) Evaluate

a) $\int_0^\infty e^{-x^2}\,dx$ ANS: $\frac{\sqrt{\pi}}{2}$

b) $\int_0^1 x^4(1-x)^2\,dx$ ANS: $\frac{1}{105}$

34) Evaluate

a) $\int_0^1 x(8-x^3)^{1/3}\,dx$ ANS: $\frac{16\pi}{9\sqrt{3}}$

b) $\int_0^{\frac{\pi}{2}} \sin^5\theta\cos^{7/2}\theta d\theta$ ANS: $\frac{64}{1989}$

c) $\int_0^\infty -3^{-4x^2}\,dx$ ANS: $\sqrt{\frac{\pi}{16\log 3}}$

35) Prove that

a) $\beta(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$ , $m>0, n>0$

b) $\int_0^1 \frac{x^{m-1}(1-x)^{n-1}}{(x+a)^{m+n}}dx = \frac{\beta(m,n)}{a^n(1+a)^m}$

c) $$\Gamma(n)\Gamma(n-1)=\frac{\pi}{\sin n\pi}$$

d) $\beta(m,n)=2\int_0^{\frac{\pi}{2}}\sin^{2m-1}\theta\cos^{2n-1}\theta d\theta$

36) Show that $\beta(m,n)=\int_0^1\frac{x^{m-1}+x^{n-1}}{(1+x)^{m+n}}dx$

37) **Prove that** (a) $=\int_0^{\infty}e^{-ky}y^{n-1}dy=\frac{\overline{|n}}{k^n}$ (b) $\overline{|1/2}=\sqrt{\pi}$

(c) $\int_0^{\infty}\left(\log 1/y\right)^{n-1}dy=\overline{|n}$

(d) $$\int_0^{\pi/2}\text{Sin}^p\theta\,\text{Cos}^q\theta=\frac{\left(\frac{p+1}{2}\right)\left(\frac{q+1}{2}\right)}{2\left(\frac{p+q+2}{2}\right)}$$

*38)* **Problem :** Find by double integration the area enclosed by the pair of curves (paraboles) *$y^2=4ax$ and $x^2=4ay$.*

*Ans:* $\frac{16}{3}a^2$

39) Find the area lying inside the circle $x^2+y^2-2ax=0$ and outside the circle $x^2+y^2=a^2$ using double integration.

ANS : $\pi\left(a^2+\frac{b^2}{2}\right).$

40) Find the volume of the tetrahedron bounded by the coordinate plane x=0 , y=0 , z=0 and the plane $\frac{x}{a}+\frac{x}{b}+\frac{x}{c}=1.$

ANS: $\frac{abc}{6}.$

41) **Problem :** Find the area between the curves *$y=2-x$ & $y^2=2(2-x)$.*

Ans: $\frac{2}{3}$

42) **Problem :** Find the volume bounded by the cylinder *$x^2+y^2 = 4$* and the planes *$y + z = 4$* and *$z = 0$.*

Ans: $16\pi$

43) **Problem :** Find the volume of the cylindrical column standing on the area common to the parabolas $y^2 = x$, $x^2 = y$ and cut off by the surface $z = 12 + y - x^2$

Ans: $\frac{569}{140}$

44) Find the volume of a sphere with centre at (a,b,c) and unit radius using spherical polar coordinate.

ANS: $\frac{4\pi}{3}$

**45) Prove that** $$\iint_D x^{l-1} y^{m-1} dxdy = \frac{\overline{|l}\,\overline{|m}}{|l+m+1} h^{l+m}$$

where D is the domain $x \geq 0,\ y \geq 0\ \ and\ \ x + y \leq h$

46) Show that $\iiint \frac{dxdydz}{(x+y+z+1)^3} = \frac{1}{2}\log 2 - \frac{5}{16}$ ,the integral being taken throughout the volume bounded by x = 0, y = 0, z = 0, x + y + z = 1.

47) **Problem :** Find the mass of an octant of the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ the density at any point being $p = kxyz$

Ans: $\frac{ka^2b^2c^2}{48}$

48) **Problem :** Evaluate $\iiint \frac{dx_1 dx_2 .......dx_n}{\sqrt{1 - x_1{}^2 - x_2{}^2 ........x_n{}^2}}$, integral being extended to all positive values of the variables for which the expression is real.

Ans: $\frac{1}{2^n} \frac{\pi^{\frac{n+1}{2}}}{\left|\frac{n+1}{2}\right.}$

49) **Problem :** Evaluate $\iiint \sqrt{\frac{1 - x^2 - y^2 - z^2}{1 + x^2 + y^2 + z^2}} dxdydz$ , integral being taken over all the values of *x, y , z* such that $x^2 + y^2 + z^2 \leq 1$

Ans: $\frac{\pi}{4}\left[B\left(-\frac{3}{4}, \frac{1}{2}\right) - B\left(\frac{5}{4}, \frac{1}{2}\right)\right]$

50) **Problem :** Evaluate $\iiint \log(x + y + z) dxdydz$, the integral extending over all positive and zero values of *x, y, z* subject to *x + y + z < 1*

Ans: $-\frac{1}{18}$.