TUTORIAL-6: B.Tech. Sem II, Mathematics-II (Fourier Series)

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Preamble: (i). $\int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx) + c$

(ii) $\int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx) + c$

(iii) $\int u\,v\ dx = uv_1 - u'v_2 + u''v_3 - u'''v_4 + ...,$ u, v are functions of x and dashes denote differentiation and suffixes denote integration with respect to x

(iv) Fourier series for $f(x)$ in $c \le x \le c + 2\pi$ is $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx,$

where $a_0 = \frac{1}{\pi}\int_c^{c+2\pi} f(x)\,dx,\ a_n = \frac{1}{\pi}\int_c^{c+2\pi} f(x)\cos nx\,dx$

$, b_n = \frac{1}{\pi}\int_c^{c+2\pi} f(x)\sin nx\,dx$

(v) Fourier series for $f(x)$ in $c \le x \le c + 2l$ is $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l},$

where $a_0 = \frac{1}{l}\int_c^{c+2l} f(x)\,dx,\ a_n = \frac{1}{l}\int_c^{c+2l} f(x)\cos\frac{n\pi x}{l}\,dx$

$, b_n = \frac{1}{l}\int_c^{c+2l} f(x)\sin\frac{n\pi x}{l}\,dx$

(vi) The half range cosine series in $0 \le x \le l$is $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}$

where $a_0 = \frac{2}{l}\int_0^l f(x)\,dx,\ a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\,dx$

, and half range sine series in $0 \le x \le l$ is $f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$ where $b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx$

Q.1 Express $f(x) = \frac{1}{2}(\pi - x)$ in a Fourier series in the interval $0 < x < 2\pi$. Deduce that $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + ...$ **Ans:**
$f(x) = \sum_{n=1}^{\infty}\frac{\sin nx}{n}$

Q.2 Find the Fourier series to represent the function $f(x) = |\sin x|,\ -\pi < x < \pi.$

Ans:
$|\sin x| = \frac{2}{\pi} - \frac{4}{\pi}\left(\frac{\cos 2x}{3} + \frac{\cos 4x}{15} + ...,\frac{\cos 2nx}{4n^2-1} + ...\right)$

Q.3 Express $f(x) = |x|$, $-\pi < x < \pi$,as a Fourier series. Hence show that

$\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+ \text{.........................} = \frac{\pi^2}{8}$. **Ans:**

$|x| = \frac{\pi}{2} - \frac{4}{\pi}\left(\cos x + \frac{cos3x}{3^2} + \frac{cos5x}{5^2} + ...\right)$

Q.4 Find the Fourier series for function Hence show that **Ans:**

$f(x) = \frac{\pi^2}{3} + 4\left(\frac{-\cos x}{1^2} + \frac{\cos 2x}{2^2} - \frac{\cos 3x}{3^2} + ...\right) - 2\left(\frac{-\sin x}{1} + \frac{\sin 2x}{2} - \frac{\sin 3x}{3} + ..\right)$

Q.5 Expand the function $f(x) = x\sin x$ as a Fourier series in the interval $-\pi \le x \le \pi$.

Ans:

$x\sin x = 1 - \frac{1}{2}\cos x - 2\left(\frac{\cos 2x}{2^2-1} - \frac{\cos 3x}{3^2-1} + \frac{\cos 4x}{4^2-1} - ...\right)$

Q.6 Find the Fourier series to represent the function $f(x) = \{$. Also deduce that $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + ...$

Ans: $f(x) = \frac{4k}{\pi}\left(\sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + ...\right)$

Q.7 Find the Fourier series of $f(x) = \{$ which is assumed to be periodic with period 2π.

Ans: $f(x) = \frac{-\pi}{2} + \frac{4}{\pi}\left(\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + ...\right) + 4\left(\frac{\sin x}{1} + \frac{\sin 3x}{3} + ...\right)$

Q.8 Obtain Fourier series for function

Ans: $\frac{\pi}{2} - \frac{4}{\pi} \le \left(\frac{\cos \pi x}{1^2} + \frac{\cos 3\pi x}{3^2} + \frac{\cos 5\pi x}{5^2} + ...\right)$

Q.9 Find the Fourier series to represent $f(x) = x^2 - 2,\ -2 \le x \le 2$.

Ans:

$x^2 - 2 = -\frac{2}{3} - \frac{16}{\pi^2}\left(\cos\frac{\pi x}{2} - \frac{1}{4}\cos \pi x + \frac{1}{9}\cos\frac{3\pi x}{2} - ...\right)$

Q.10 Obtain the half range sine series for $f(x) = e^x,\ 0 < x < 1$. **Ans:**

$e^x = 2\pi \sum_{n=1}^{\infty} \frac{n[1-e(-1)^n}{1+n^2\pi^2} sin\, n\pi x$

Q.11 Express $\sin x$ as a cosine series in $0 < x < \pi$.

Ans: $\sin x = \frac{2}{\pi} - \frac{4}{\pi}\left(\frac{\cos 2x}{1.3} + \frac{\cos 4x}{3.5} + \frac{\cos 6x}{5.7} + ...\right)$

Q.12 Obtain the half-range sine series for the function $f(x) = x^2$ in the interval $(0, 3)$.

Ans: $f(x) = \sum_{n=1}^{\infty} b_n \frac{\sin n\pi x}{3},\ where\ b_n = \frac{18}{n\pi}(-1)^{n+1} + \frac{36}{n^3\pi^3}\left[(-1)^n - 1\right]$