Tutorial–1, B. Tech. Sem II, 5 Sep., 2022
(Introduction of Complex numbers & Analytic functions)

1. Find the locus of z in each of the following relations: $(i)|z-5| = 6$, (ii)$|z+2i| \geq 1$, (iii)$Re(z+2) = -1$, (iv)$|z-i| = |z+i|$, (v)$|z+3|+|z+1| = 4$, (vi)$1 \leq |z-3| \leq 2$, (vii) $|z+3|-|z+1| = 1$.
2. For which complex number following are true, justify in each (i)$z = -z$, (ii)$-z = z^{-1}$, (iii) $z = z^{-1}$, (iv)$z = \bar{z}$
3. Define an analytic function at a point and in a domain.
4. Prove that an analytic function of constant modulus is always constant.
5. Prove that Real and imaginary parts of an analytic function are harmonic.
6. Prove that an analytic function is always continuous but converse need not be true. Give an example.
7. State and prove the necessary and sufficient condition for a function $f(z) = u + iv$ to be analytic.
8. Define an analytic function at a point. Illustrate such a function.
9. If $f(z) = \frac{(\bar{z})^2}{z}, z \neq 0; f(0) = 0$ then $f(z)$ satisfies Cauchy–Riemann equations (CR) at origin.
10. Using Milne–Thomson method construct an analytic function $f(z) = u + iv$ for which $2u + 3v = 13(x^2 - y^2) + 2x + 3y$.
11. State and Prove Cauchy–Riemann equations in polar coordinate system.
12. Let $f(x,y) = \sqrt{(|xy|)}$, then (a) f_x, f_y do not exist at $(0,0)$; (b) $f_x(0,0) = 1$; (c) $f_y(0,0) = 0$; (d) f is differentiable at $(0,0)$. **Ans**: c
13. If function $f(z) = \left(\frac{\bar{z}}{z}\right)^2$, when $z \neq 0$, $f(z) = 0$ for $z = 0$, Then $f(z)$ (a) satisfies C.R. equations at $z = 0$; (b) is not continuous at $z = 0$; (c) is differentiable at $z = 0$; (d) is analytic at $z = 1$ **Ans**: b
14. Show that $f(z) = \frac{(\bar{z})^2}{z}$, when $z \neq 0$, $f(z) = 0$ for $z = 0$, satisfies C.R. equations at $z = 0$; but is not differentiable at $z = 0$.
15. The harmonic conjugate of $u = x^2 - y^2 + xy$ is (a) $x^2 - y^2 - xy$; (b) $x^2 + y^2 - xy$; (c) $1/2(-x^2 + y^2) + 2xy$. (d) $2(-x^2 + y^2) + 1/2$ Ans: c

16. $f(z) = (|z|)^2$ is (a) continuous everywhere but nowhere differentiable; (b) continuous at $z = 0$ but differentiable everywhere; (c) continuous nowhere; (d) none of these. **Ans**: d

17. Examine the nature of the function $f(z) = \frac{x^2y^5(x+iy)}{x^4+y^{10}}$; if $z \neq 0$, otherwise 0, in a region including the origin.

18. Prove that the function $f(z) = u + iv$, where $f(z) = \frac{x^3(1+i)-y^3(1-i)}{x^2+y^2}$; if $z \neq 0$, otherwise 0 is continuous and that Cauchy–Riemann equations are satisfied at the origin, yet $f'(z)$ does not exists there.

19. If $f(z) = \frac{x^3y(y-ix)}{x^6+y^2}$; if $z \neq 0$ and $f(0) = 0$, show that $\frac{f(z)-f(0)}{z} \to 0$ as $z \to 0$ along any radius vector but not as $z \to 0$ in any manner.

20. If $f(z) = \frac{xy^2(x+iy)}{x^2+y^4}$; if $z \neq 0$ and $f(0) = 0$, prove that $\frac{f(z)-f(0)}{z} \to 0$ as $z \to 0$ along any radius vector but not as $z \to 0$ in any manner.

21. The function $f(z) = z$ is (a) analytic at $z = 0$, (b) differentiable only at $z = 0$;(c) satisfies C.R. equations everywhere; (d) nowhere analytic. **Ans: d**

22. Derive the C.R. equations for an analytic function $f(r, \theta) = u(r, \theta) + i\, v(r, \theta)$ and deduce that $u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0$.

23. Find the point where the C.R. equations are satisfied for the function $f(z) = xy^2 + ix^2y$. In which region $f'(z)$ exists?

24. $f(z) = (|z|)^2$ is (a) continuous everywhere but nowhere differentiable except at 0; (b) continuous at $z = 0$ but differentiable everywhere (c) continuous nowhere; (d) none of these. **Ans**: a

25. Prove that the function $f(z) = z|z|$ is nowhere analytic.

26. If $f(z)$ is an analytic function such that $Re f'(z) = 3x^2 - 4y - 3y^2$ and $f(1 + i) = 0$ then $f(z)$ is (a) $z^3 + 6 - 2i$, (b) $z^3 + 2iz^2 + 6 - 2i$, (c) $z^3 + 2iz^2 - 2i$, (d) $z^3 + 2z^2 + 6 - 2i$ **Ans**: b
Hint: $u_x = 3x^2 - 4y - 3y^2 = \phi_1(x, y)(say)$, integrating partially w.r.t. y we get $u = x^3 - 4xy - 3xy^2 + g(y)$
Therefore, $u_y = -4x - 6xy + g'(y)$
or $-u_y = v_x = \phi_2(x, y)(say) = 4x + 6xy - g'(y)$
Thus $\phi_1(z, 0) = 3z^2, \phi_2(z, 0) = 4z + g'(0)$
Now, applying Milne Thomson we get
$f(z) = \int(3z^2 + i\,4z)dz + constt.$
or $f(z) = z^3 + 2i\,z^2 + constt.$ and applying $f(1 + i) = 0$ implies $Constt = 6 - 2i$

27. The orthogonal trajectory of $u = e^x(x\cos y - y\sin y)$ is (a) $e^x(\cos y + x\sin y) + c$; (b) $e^x x\sin y + c$; (d) $e^x(y\cos y + x\sin y) + c$. **Ans**: c

28. Find the locus of points in the plane satisfying the relation $|z + 5|^2 + |z - 5|^2 = 75$. **Ans**: circle

29. The function $f(z) = \bar{z}$ is (a) analytic at $z = 0$, (b) continuous at $z = 0$;(c) differentiable only at $z = 0$; (d) analytic anywhere. **Ans**: b

30. If $f(z) = u + iv$ is an analytic function and $u - v = (x - y)(x^2 + 4xy + y^2)$ then $f(z)$ is (a)$-z^3 + c$, (b) $-iz^2 + ic$, (c)$-iz^3 + \beta$, (d) $z^3 - ic$, **Ans**: c

31. If $f(z) = u + iv$, is an analytic function of z and $u - v = \frac{\cos x + \sin x - e^{-y}}{2\cos x - e^{y} - e^{-y}}$; find $f(z)$ if $f(z)$ subject to the condition $f(\pi/2) = 0$. **Ans**: $1/2\{1 - \cot(z/2)\}$

32. If $f(z) = u(r, \theta) + iv(r, \theta)$ is an analytic function and $u = -r^3 \sin 3\theta$ then construct the analytic function $f(z)$.

33. If $f(z) = u + iv$, is analytic function of z and $u + v = \frac{2\sin 2x}{-2\cos 2x + e^{2y} - e^{-2y}}$; find $f(z)$ in terms of z. **Ans**: $\frac{1}{2}(1 + i)\cot z + d$

34. Choose the correct code for matching list A and B.

A (u is given)	B ($f(z) = u + iv$ is an analytic function)
p. $x^3 - 3xy^2 + 3x + 1$	(i). $\sin z + ci$
q. $y^3 - 3x^2y$	(ii).$z^3 + 3z + 1 + ci$
r. $\sin x \cosh y$	(iii).$i(z^3 + c)$

i ii iii
(a) p r q
(b) r p q
(c) p q r **Ans**: b

35. if $\sin(\alpha + i\beta) = x + iy$ prove that (a) $\frac{x^2}{\cosh^2\beta} + \frac{y^2}{\sinh^2\beta} = 1$, (b) $\frac{x^2}{\cos^2\alpha} - \frac{y^2}{\sin^2\alpha} = 1$.

36. Prove that $(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2})|f(z)|^2 = 4|f'(z)|^2$ for an analytic function $f(z) = u + iv$.

37. Find the harmonic conjugate of $u = x^3 - 3xy^2 + 3x^2 - 3y^2 + 2x + 1$ and corresponding analytic function $f(z) = u + iv$. **Ans**: $v = 3x^2y - y^3 + 6xy + 2y + d$, $f(z) = z^3 + 3z^2 + 2z + 1 + id$.

38. Find orthogonal trajectory of $v = e^{2x}(x\cos 2y - y\sin 2y)$ **Ans**: $-e^{2x}(x\sin 2y + y\cos 2y) + d$

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Tutorial–2, B. Tech. Sem II, 21 August, 2022

(Complex Integrations)

1. Define simply and multiply connected regions? State and prove Cauchys theorem for an analytic function? Is it true for multiply connected regions?

2. State and prove Cauchy integral formula for nth derivative? Is it true for multiply connected regions? If yes, give your explanation with necessary proofs.

3. If $f(z)$ is analytic in a simply connected region D and a, z are two points in D, then $\int\limits_a^z f(z)dz$ is independent of the path in D joining a and z.

4. Evaluate $\int_{(0,3)}^{(2,4)} \bar{z}\,dz$ along (a) parabola $x = 2t, y = t^2 + 3$, (b) a straight line joining $(0,3)$ and $(2,4)$. Find whether both values are different, if yes, justify reason why it is so?

5. Evaluate $\int_C \bar{z}\,dz$ from $z = 0$ to $z = 4 + 2i$ along the curve C given by $(i) z = t^2 + it$; (ii) the line from $z = 0$ to $z = 2i$ and then the line from $z = 2i$ to $z = 4 + 2i$. **Ans:** (i) $10 - \frac{8i}{3}$; $(ii) 10 - 8i$.

6. If $F(a) = \oint\limits_C \frac{z^2+2z-5}{z-a} dz$, where C is the ellipse $(x/2)^2 + (y/3)^2 = 1$. Find the value of $F(4.5)$. **Ans:** Zero

7. Evaluate $\oint\limits_C \frac{1}{z-a} dz$, where C is any simple closed curve and $z = a$ is (i) outside C; (ii) inside C. **Ans:** (i) 0 ; (ii) $2\pi i$.

8. Evaluate $\oint\limits_C \frac{1}{(z-a)^n} dz$; $n \neq 1$, $n \in Z^+$, where C is any simple closed curve and $z = a$ is inside C. **Ans:** Zero

9. Suppose $f(z)$ is analytic inside and on a simple closed curve C and $z = a$ is inside C. Prove that $\oint\limits_C \frac{1}{z-a} dz = \oint\limits_{C_1} \frac{1}{z-a} dz$ where C_1 is a circle center at a and totally contained in simple closed curve C.

10. Evaluate $\oint\limits_C (\bar{z})^2 dz$; where C is the circle $(i)|z - 1| = 1$, $(ii)|z| = 1$ **Ans:** $4\pi i$, 0.

11. Find $\oint\limits_C \bar{z} dz$ around (a) the circle $|z - 2| = 3$, (b) the ellipse $|z - 3| + |z + 3| = 10$, (c) the square with vertices 0, 2, $2i$ and $2 + 2i$. **Ans:** $18\pi i$, $40\pi i$, $8i$

12. Suppose $f(z)$ is integrable along a curve C having finite length l and there exists a positive real number M such that $|f(z)| \leq M$ on C. Prove that $|\int_C f(z)dz| \leq Ml$.
    Remark: This result is helpful to evaluate the upper bound of an integral without evaluating it.

13. Work out the following integrals around the contour prescribed against it
    (i) $\oint\limits_C \frac{\cos \pi z^2 + \sin \pi z^2}{(z-1)(z-2)} dz$; C: $|z| = 3$. **Ans:** $4\pi i$.
    (ii) $\oint\limits_C \frac{1}{z^2+16} dz$; C: $|z| = 6$. **Ans:** 0

(iii) $\oint\limits_C \frac{e^{5z}}{z-2i}dz$; C: $|z-2|+|z+2|=6$. **Ans:** $2\pi i e^{10i}$

(iv) $\frac{1}{2\pi i}\oint\limits_C \frac{e^z}{z-2}dz$; C: $|z|=3$. **Ans:** e^2

(v) $\oint\limits_C \frac{\sin \pi z}{z^2-1}dz$;

- C is the rectangle with vertices $2+i$, $2-i$, $-2+i$, $-2-i$;
- C is the rectangle with vertices $-i$, $2-i$, $2+i$, i. **Ans:** 0, 0.

(vi) $\oint\limits_C \frac{e^{\pi i z}}{z^2-4z+5}dz$; C: $|z-1-2i|=2$. **Ans:** $\pi e^{i\pi}$

(vii) $\oint\limits_C \frac{e^z+\cos z}{(z-5)(z+5i)}dz$; where C is the boundary of a triangle with vertices: -1, 1, $-7/2i$.
Ans: $\frac{2\pi i(\pi i-5)}{\pi^2+25}(2\cos \pi i - \sin \pi i)$

(viii) $\oint\limits_C \frac{\sin^6 z}{z-\pi/6}dz$; C: $|z|=2$. **Ans:** $\pi i 2^{-5}$

(ix) Show that $\int_C \frac{e^{z^2}}{z^2(z-1-i)}dz = \pi e^{2i}$; where C consists of $|z|=2$ anticlockwise and $|z|=1$ clockwise.

(x) $\frac{1}{2\pi i}\oint\limits_{C:|z|=4} \frac{e^z}{(z+2)^2}dz$; **Ans:** e^{-2}

14. Evaluate $\oint\limits_C \frac{1}{z^2-1}dz$ around $z=-i+5e^{it}$. **Ans:** 0

15. Evaluate $\oint\limits_C (z^2+3+4/z)dz$, C: $|z|=4$. **Ans:** $8\pi i$

16. Evaluate the upper bound of the integral without evaluating it ?

    (a) $\int_C (z^4+1)dz$, C : Line segment from 0 to $1+i$. **Ans:** $5\sqrt{2}$.

    (b) $\int_C z dz$, C : Line segment from 0 to i. **Ans:** 1.

    (c) $\int_C 2z dz$, C : Line segment from i to $2+i$. **Ans:** $4\sqrt{5}$.

    (d) $\int_C (x^2+iy^2)dz$, C : $z=e^{it}$, $-\pi/2 < t < \pi/2$. **Ans:** π

17. Which of the following integrals are compatible to apply the Cauchy theorem?

1. $\oint\limits_C \frac{sinz}{z+2i}\, dz$, C : $|z|=1$.

2. $\oint\limits_C \frac{sinz}{z+2i}\, dz$, C : $|z+3i|=1$.

3. $\oint\limits_C e^{\bar{z}}\, dz$, C : $|z-3i|=6$.

4. $\oint\limits_{|z|=b} \frac{1}{z^2+bz+1}\, dz$, $0<b<1$.

5. $\oint\limits_{|z|=3} \frac{1}{1-e^z}\, dz$.

6. $\int\limits_{0}^{1+i} z^3\, dz$, along $y = x$.

1. Evaluate $\oint\limits_{|z|=3} \frac{1}{z^3-z}\, dz$, **Ans:** 0

2. Evaluate $\oint\limits_{C} \frac{z^2}{z-2}\, dz$ where C is the boundary of a triangle with vertices -1, 0 and $2i$. **Ans:** 0

3. Evaluate the following integrals around the contour prescribed against it
   (i) $\oint\limits_{C} \frac{z+e^z}{(z+i\pi)^3} dz$; C: $z = 7e^{it}$, $0 \le t \le 2\pi$. **Ans:** $-\pi i$.
   (ii) $\oint\limits_{C} \frac{1}{z^3(z-2)^2} dz$; C: $|z-3| = 2$, **Ans:** $\frac{-3\pi i}{8}$.
   (iii) $\oint\limits_{C} \frac{z+e^z}{(z+i\pi)^3} dz$; C: $z = 7e^{it}$, $0 \le t \le 2\pi$. **Ans:** $-\pi i$.
   (iv) $\oint\limits_{C} \frac{1}{z^3(z-2)^3} dz$; C: $|z-1| = 3$, **Ans:** 0.
   (v) $\frac{1}{2\pi i} \oint\limits_{|z|=10} \frac{3z^4}{(z-6i)} dz$. **Ans:** 3×6^4
   (vi) $\oint\limits_{|z-1|=5} \frac{1}{z^4-1} dz$. **Ans:** 0.
   (vii) $\oint\limits_{|z-i|=3/2} \frac{1}{z^4(z+i)} dz$. **Ans:** $-4\pi i$.
   (viii) $\oint\limits_{|z-1|=3} \frac{e^z-z^2}{(z-2)^3} dz$. **Ans:** $2\pi i(2e^4-1)$;
   (ix) $\oint\limits_{|z|=2} \frac{\sin z}{(z-1)^2} dz$. **Ans:** $2\pi i \cos 1$;
   (x) $\oint\limits_{|z+i|=3/2} \frac{\cos z}{(z+3i)^6} dz$. **Ans:** 0;
   (xi) $\oint\limits_{C} \frac{3}{z^2(z+i)^2} dz$; C: $|z| = 5$, **Ans:** 0.
   (xii) $\oint\limits_{|z|=2a} \frac{e^z}{z^2+a^2} dz$. **Ans:** $\frac{2\pi i}{a} \sin a$;
   (xiii) $\oint\limits_{|z-ia|=a} \frac{e^z}{z^2+a^2} dz$. **Ans:** $\frac{e^{ia}\pi}{a}$
   (xiv) $\oint\limits_{C} \frac{z+1-e^z}{z(z+3)} dz$; C: $|z-i| = 2$, **Ans:** 0.

4. Evaluate $\oint\limits_{C} (x^2+iy^2) ds$; C: $|z| = 2$ where s is the arc length. **Ans:** $8\pi(1+i)$

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Tutorial–3, B. Tech. Sem II, 5 Sept., 2022

(Laurent's Expansions)

1. State and prove Laurents series expansion of a function $f(z)$.

2. Find Laurents series expansion about the indicated singularity for each of the following functions. Name the singularity in each case and give the region of convergence of each series.

   (i). $\frac{e^{2z}}{(z-1)^3}$; $z = 1$

   Ans: $e^2[\frac{1}{(z-1)^3} + \frac{2}{(z-1)^2} + \frac{2}{(z-1)} + \frac{4}{3} + \frac{2(z-1)}{3} + ...$; $z = 1$ is a pole of order 3 and Series converges for all $z \neq 1$.

   (ii). $(z-3)\sin\frac{1}{(z+2)}$; $z = -2$;

   Ans: $1 - \frac{5}{z+2} - \frac{1}{6(z+2)^2} + \frac{5}{6(z+2)^3} + \frac{1}{120(z+2)^4} - ....$; $z = -2$ is an essential singularity and Series converges for all $z \neq -2$.

   (iii). $\frac{z-sinz}{Z^3}$; $z = 0$;

   Ans: $\frac{1}{3!} - \frac{z^2}{5!} - \frac{z^4}{7!}..$; $z = 0$ is a removable singularity. Series converges for all z.

   (iv). $\frac{z}{(z+1)(z+2)}$;$z = -2$;

   Ans: $\frac{2}{z+2} + 1 + (z+2) + (z+2)^2 + ..$; $z = -2$ a simple pole and Series converges for all z in $0 < |z+2| < 1$.

   (v). $\frac{1}{z^2(z-3)^2}$; $z = 3$

   Ans: $\frac{1}{9(z-3)^2} - \frac{2}{27(z-3)} + \frac{1}{27} - \frac{(z-3)}{243} + ...$; $z = 3$ is a pole of order 2 and Series converges for all z in $0 < |z-3| < 3$.

3. Expand $f(z) = \frac{1}{(z+1)(z+3)}$ in a Laurent' s series valid for expansion (a). $1 < |z| < 3$, (b). $|z| > 3$, (c) $0 < |z+1| < 2$, (d) $|z| < 1$,

   Ans: (a). $\cdots\cdots - \frac{1}{2z^4} + \frac{1}{2z^3} - \frac{1}{2z^2} + \frac{1}{2z} - \frac{1}{6} + \frac{z}{18} - \frac{z^2}{54} + \frac{z^3}{162} - \cdots\cdots$;

   (b). $\frac{1}{z^2} - \frac{4}{z^3} + \frac{13}{z^4} - \frac{40}{z^5} + \cdots\cdots$;

   (c). $\frac{1}{2(z+1)} - \frac{1}{4} + \frac{(z+1)}{8} - \frac{1}{16}(z+1)^2 + \cdots\cdots$;

   (d). $\frac{1}{3} - \frac{4z}{9} + \frac{13}{27}z^2 + \cdots\cdots\cdots\cdots$.

4. Expand $f(z) = \frac{z}{(z-1)(2-z)}$ in a Laurents series valid for (a). $1 < |z| < 2$; (b). $|z| > 2$; (c). $0 < |z-2| < 1$; (d). $|z| < 1$; (e). $|z-1| > 1$.

   Ans: (a). $1 + \frac{z}{2} + \frac{z^2}{4} + \frac{z^3}{8} + \cdots\cdots \frac{1}{z} + \frac{1}{z^2} + \cdots\cdots$;

   (b). $-\frac{1}{2} - \frac{3}{z^2} - \frac{7}{z^3} - \frac{15}{z^4}\cdots\cdots$;

   (c). $1 - \frac{2}{z-2} - (z-2) + (z-2)^2 - (z-2)^3 + (z-2)^4\cdots\cdots$;

   (d). $-\frac{1}{2}z - \frac{3}{4}z^2 - \frac{7}{8}z^3 - \frac{15}{19}z^4 - \cdots\cdots$.

   (e). $-\frac{1}{(z-1)} - \frac{2}{(z-1)^2} - \frac{2}{(z-1)^3}\cdots\cdots$;

5. Expand $f(z) = \frac{1}{z-3}$ in a Laurents series valid for (a). $|z| < 3$; (b). $|z| > 3$

   Ans: (a). (i)$-1/3 - 1/9z - 1/27z^2 - 1/81z^3 - \cdots\cdots$;

   (ii) $z^{-1} + 3z^{-2} + 9z^{-3} + 27z^{-4} + \cdots\cdots$.

6. Expand $f(z) = \frac{1}{z(z-2)}$ in a Laurent's series valid for (a). $0 < |z| < 2$; (b). $|z| > 2$

   Ans: (a). $-\frac{1}{2z} - \sum_{n=0}^{\infty}\frac{z^n}{2^{n+2}}$; (b). $\sum_{n=1}^{\infty}\frac{2^{n-1}}{z^{n+1}}$;

7. Find Taylor's series expansion about the indicated points for each of the following functions. Give the region of convergence of each series.
(i). $\frac{z}{e^z+1}$; $z = 0$;
(ii). $\frac{\sin z}{z^2+4}$; $z = 0$
Ans: (i) $|z| < \pi$; (ii) $|z| < 2$

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Tutorial–4, B. Tech. Sem II, 10 Sep, 2022

(Singularities and Residues)

1. Discuss the types of each singularity for the following functions:
   (i). $\frac{1}{\cos z - \sin z}$; $z = \pi/4$ is a simple pole.
   (ii) . $\frac{\cot \pi z}{(z-a)^2}$; $z = a$ is a pole of order 2.
   (iii). $\frac{e^z}{z^2+1}$; $z = \pm i$ is a simple pole and $z = \infty$ is an essential singularity.
   (iv). $\frac{z}{(z+1)(z+2)}$; $z = -1, -2$ are simple poles.
   (v). $\frac{e^z}{\sin z}$; $z = k\pi, k \in \mathbb{I}$ is a simple pole.
   (vi). $\arcsin \frac{1}{z}$; $z = 1/n\pi, (n \in \mathbb{I}^+)$ simple pole and $z = 0$ an isolated essential singularity.
   (vii). $\arctan \pi z$; $z = 1/n, (n \in \mathbb{I}^+)$ is simple pole and $z = 0$ isolated essential singularity.
   (viii). $\frac{z - sinz}{z^3}$; $z = 0$ is a removable singularity.
   (ix). $\frac{z+1}{z^2-2z}$; $z = 0, 2$ are simple poles.
   (x). $\frac{z^2}{(z-1)^2(z+2)}$; $z = -2$ is a simple pole and $z = 1$ is a pole of order 2.
   (xi). $\frac{1}{z^4+1}$; $e^{\frac{(i\pi+2k\pi)}{4}}, \forall k = 0, 1, 2, 3, 4$ each is a simple pole.
   (xii). $\frac{ze^{iz}}{z^2+a^2}$; $z = \pm ai$ are simple poles.
   (xiii). $\frac{1-e^{2z}}{z^3}$; $z = 0$, a pole of order 2.
   (xiv). $\frac{e^{2z}}{1+e^z}$; $z = i\pi$ is a simple pole.
   (xv). $\frac{e^{2z}}{1-e^z}$; $z = 0$ is a simple pole.
   (xvi). $\frac{z - sinz}{z^5}$; $z = 0$ is a pole of order 2.
   (xvii). $\frac{1}{z^2(e^z-1)}$; $z = 0$ is a pole of order 3.
   (xviii). $\frac{\sinh z}{z^2}$; $z = 0$ is a simple pole.
   (xix). $\frac{(z^2+1)e^z}{(z-1)^3(z-i)}$; $z = 1$ is a pole of order 3 and $z = i$ is a removable singularity.

2. Examine the nature of singularity for the following functions and find its residue:
   Hint: residue is defined as coefficient of $\frac{1}{(z-z_0)}$ in Laurent's series expansion of a function $f(z)$ about a point z_0. Thus,
   residue for a pole of order m is $\frac{1}{(m-1)!} \lim_{z \to z_0} \frac{d^{m-1}}{dz^{m-1}}((z - z_0)^m f(z))$
   residue for a removable singularity=0.
   residue for an essential singularity z_0 can be find by finding coefficient of $\frac{1}{z-z_0}$ in its Laurent's series expansion about a point z_0.
   (i). $\frac{e^z+\sin z}{z^4}$; $z = 0$ is a removable singularity, Residue= 0
   (ii). $\frac{1}{(z+i)^3}$; $z = i$ is a a pole of order 3. Residue $= \frac{3}{6i}$.
   (iii). $\frac{ze^{iz}}{z^2+a^2}$; $z = \pm ai$ are simple poles. Residues are $= \frac{e^{-a}}{2}$. and= $\frac{e^a}{2}$.
   (iv). $\frac{1}{z^2 \sinh z}$; $z = 0$ a pole of order 3. Residue= $-\frac{1}{6}$

3. Using Residue theorem evaluate the following integrals:
   (i). $\int_0^{2\pi} \frac{1}{2+\cos\theta} d\theta$, **Ans:** $\frac{2\pi}{\sqrt{3}}$.
   (ii). $\int_0^{2\pi} \frac{1}{1-2a\cos\theta+a^2} d\theta$, $a > 1$ **Ans:** ....
   (iii). $\int_0^{\pi} \frac{1+2\cos\theta}{5+3\cos\theta} d\theta$, **Ans:** $\frac{\pi}{24}$.
   (iv). $\int_0^{2\pi} \frac{\cos 2\theta}{5+4\cos\theta} d\theta$, **Ans:** $\frac{\pi}{6}$
   (v). $\int_0^{2\pi} \frac{\cos^2 3\theta}{5-4\cos 2\theta} d\theta$, **Ans:** $\frac{3\pi}{8}$